Sketchy solutions of the midterm in Comlex.Functions.EE
(201.1.0071) 28.05.2017 Ben Gurion University

(1) Note that for $|z|<1$ we have: $1+\frac{z}{3}\notin\mathbb{R}_{<0}$, thus $log(1+\frac{z}{3})\in\mathcal{O}(D_1(0))$.

Solution 1. $\oint\limits_{|z|=1}\frac{log(1+\frac{z}{3})}{z^3}dz=\sum\limits_{n=1}^{\infty}\oint\limits_{|z|=1}-\frac{(-\frac{z}{3})^n}{n}\frac{dz}{z^3}=-\frac{\pi i}{9}$.

Here we are allowed to change the integration and the summation because the series converges uniformly in $\overline{D_1(0)}$. (And the series converges uniformly in $\overline{D_1(0)}$, e.g. because it converges in $D_3(0)$.)

Solution 2. By the integral presentation of Cauchy (for derivatives) we have:

$$\oint\limits_{|z|=1}\frac{log(1+\frac{z}{3})}{z^3}dz=\frac{2\pi i}{2}\Big(log(1+\frac{z}{3})\Big)''|_{z=0}=-\frac{\pi i}{9(1+\frac{z}{3})^2}|_{z=0}=-\frac{\pi i}{9}.$$

(2) Solution 1. The denominator vanishes at the points $0,\pm\pi$. These are the only points where f could be non-holomorphic. But the limits $\lim\limits_{z\to0}f(z)$, $\lim\limits_{z\to\pi}f(z)$, $\lim\limits_{z\to-\pi}f(z)$ exist and are finite. Therefore the points $0,\pm\pi$ are removable singularities, thus f extends to a holomorphic function on the whole $\mathbb{C}$. Thus, by Cauchy theorem, the primitive function of f exists on the whole $\mathbb{C}$. And in particular on $\mathbb{C}\setminus D_{10}(0)$.

Solution 2. To prove the existence of a primitive function, it is enough to check the vanishing $\oint\limits_{\gamma}f(z)dz=0$ for any closed curve γ. The function is obviously holomorphic in $\mathbb{C}\setminus D_{10}(0)$, thus the only integral to check is $\oint\limits_{|z|=10}\frac{sin(z)}{z(z^2-\pi^2)}dz$. Expand the fraction: $\frac{1}{z(z^2-\pi^2)}=\frac{a}{z}+\frac{b}{z-\pi}+\frac{c}{z+\pi}$, with some coefficients $a,b,c\in\mathbb{C}$. Note: $\oint\limits_{|z|=10}\frac{sin(z)}{z}dz\overset{Cauchy}{=\!=}2\pi i\cdot sin(0)=0$. Similarly $\oint\limits_{|z|=10}\frac{sin(z)}{z\pm\pi}dz=sin(\pm\pi)=0$. Thus $\oint\limits_{|z|=10}\frac{sin(z)}{z(z^2-\pi^2)}dz=0$. Therefore (e.g. by Morrera theorem) there exists a primitive function of f, on $\mathbb{C}\setminus D_{10}(0)$.

(3) As $f(0)=0$ but does not vanish identically, the zero of f at $z=0$ is of a finite order. Thus $f(z)=z^kg(z)$, for some $k\in\mathbb{N}$ and $g(z)\in\mathcal{O}(\mathbb{C})$ with $g(0)\neq0$. Now, the holomorphic function g satisfies: $|g|_{|z|=1}=1$. Thus, by the argument of maximum, $|g(z)|\le1$ on $D_1(0)$. But g does not vanish on $D_1(0)$, thus $\frac{1}{g}\in\mathcal{O}(D_1(0))$. Again, the argument of maximum gives: $|\frac{1}{g(z)}|\le1$ on $D_1(0)$. Therefore $|g(z)|\equiv1$ on $D_1(0)$. As g is holomorphic, this implies: $g(z)=const$ on $D_1(0)$.

Remarks i. Many students thought that any function satisfying $|f(0)|=0$, $|f|_{|z|=1}=1$, must be of the form $f(z)=cz$. ("by Schwartz lemma") Some thought that just having $|f(1)|=1$ would suffice.
ii. To remind: there is no meaning to "$w<z$" for complex numbers.

(4) f is holomorphic except possibly at the following points: $z=100$ or $\{sin(z)=2\pi ik\}_{k\in\mathbb{Z}}$. By the direct check:
(a) The points where $sin(z)=0$ are removable singularities ($\lim\limits_{z\to\pi n}f(z)$ exists and is finite).
(b) The points where $sin(z)=2\pi ik\neq0$ are (simple) poles.
(c) The point $z=100$ is an essential singularity.
Thus f extends to a holomorphic function on $\mathbb{C}\setminus\Big\{100,\{sin(z)=2\pi ik\}_{0\neq k\in\mathbb{Z}}\Big\}$. Let z_0 be the point among $\{sin(z)=2\pi ik\}_{0\neq k\in\mathbb{Z}}$ which is the closest to 2. Then the radius of convergence is $|z_0-2|$.

Remark Most students overlooked the points $\{sin(z)=2\pi ik\}_{0\neq k\in\mathbb{Z}}$. Those who overlooked these points, but *treated carefully all the rest* (identified the (non-)removable points) have lost just one point out of 20.

To find z_0 we consider the equation $\frac{e^{iz}-e^{-iz}}{2i}=2\pi ki$. Thus $Re(z)\in\pi\mathbb{Z}$, while $t=Im(z)$ satisfies: $cos(Re(z))(e^t-e^{-t})=4\pi k$. As z_0 is the closest point to 2, we have: $Re(z_0)=\pi$, while $t=Im(z_0)$ satisfies: $e^t-e^{-t}=-4\pi$. Therefore $e^t=-2\pi+\sqrt{4\pi^2+1}$. Finally, the radius of convergence is: $|it-2|$.

(5) Solution 1. As f is holomorphic, it is inifitely differentiable at each point. Thus the derivative can be computed in various ways, e.g. $f'(z_0) = \lim\limits_{\substack{z_1,z_2\to z_0\\ z_1\neq z_2}} \frac{f(z_1)-f(z_2)}{z_1-z_2}$. By the assumption the expression $\frac{f(z_1)-f(z_2)}{z_1-z_2}$ vanishes at some points in any neighborhood of z_0. Therefore (as the limit exists) $\lim\limits_{\substack{z_1,z_2\to z_0\\ z_1\neq z_2}} \frac{f(z_1)-f(z_2)}{z_1-z_2} = 0$. Thus $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$.

Solution 2. As f is holomorphic, it is analytic at each point. Take the Taylor expansion of f at $z = z_0$:

$$f(z) = f(z_0) + f'|_{z_0}(z - z_0) + \sum_{n=2}^{\infty}(z - z_0)^n \frac{f^{(n)}|_{z_0}}{n!}.$$

Then $f(z_1) - f(z_2) = (z_1 - z_2)\Big(f'|_{z_0} + \sum\limits_{n=2}^{\infty} \frac{f^{(n)}|_{z_0}}{n!} \frac{(z_1-z_0)^n-(z_2-z_0)^n}{z_1-z_2}\Big)$.

Using the assumption, for any ϵ choose $z_1, z_2 \in D_\epsilon(z_0)$ such that $f(z_1) = f(z_2)$. Then we get: $f'|_{z_0} + \sum\limits_{n=2}^{\infty} \frac{f^{(n)}|_{z_0}}{n!} \frac{(z_1-z_0)^n-(z_2-z_0)^n}{z_1-z_2} = 0$.

Finally, note that for $n \geq 2$ holds: $\lim\limits_{\epsilon\to 0} \frac{(z_1-z_0)^n-(z_2-z_0)^n}{z_1-z_2} = 0$. (e.g. open the brackets)

As the series converges uniformly (and ϵ can be chosen arbitrarily small), we get:

$$f'|_{z_0} = -\lim_{\epsilon\to 0}\sum_{n=2}^{\infty} \frac{f^{(n)}|_{z_0}}{n!} \frac{(z_1 - z_0)^n - (z_2 - z_0)^n}{z_1 - z_2} = -\sum_{n=2}^{\infty} \lim_{\epsilon\to 0} \frac{f^{(n)}|_{z_0}}{n!} \frac{(z_1 - z_0)^n - (z_2 - z_0)^n}{z_1 - z_2} = 0.$$

Therefore f' vanishes at each point of $\mathbb{C}$, thus $f = const$.

Remarks i. In the case of real valued functions the solution is much shorter. By Lagrange's theorem, there exists $c \in [z_1, z_2]$ satisfying $f'(c) = \frac{f(z_1)-f(z_2)}{z_1-z_2} = 0$. Thus f' has zeros in any neighborhood of any point, thus $f' \equiv 0$.

ii. Note that the condition $f(z_1) = f(z_2)$ is given for pairs of points. This does not imply any sequence $\{z_n\}$ such that $f(z_n) = f(z_{n+1})$.