Partial solutions of moed.A, Comlex.Functions.EE
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(1) (This is close to problem 2 of tutorial 7, to question 5.d of hwk 6, to question 4.h of hwk 9, to question 8 of hwk 10, to question 5 of hwk 11.)

Solution 1. Note that $\{f(\frac{1}{2n})=0\}$. Thus the holomorphic function f vanishes on a (infinite) sequence of points that converges inside $D_1(0)$. Therefore, by the uniqueness principle, $f(z)\equiv 0$ on $D_1(0)$. But this contradicts the condition $f(\frac{1}{2n+1})=\frac{(-1)^{n+1}}{2n+1}\neq 0$. Therefore the function with the prescribed conditions does not exist.

Solution 2. Suppose such a holomorphic function exists, then in particular f is continuous, thus $f(0)=\lim\limits_{n\to\infty}f(\frac{1}{n})=0$. But then f cannot be $\mathbb{R}$-differentiable at 0, because $\lim\limits_{n\to\infty}\frac{|f(0)-f(\frac{1}{2n+1})|}{|0-\frac{1}{2n+1}|}=1$, while $\lim\limits_{n\to\infty}\frac{|f(0)-f(\frac{1}{2n})|}{|0-\frac{1}{2n}|}=0$. Therefore the function with the prescribed conditions does not exist.

(2) (This is close to problem 4 of hwk8.)

Note: $f^{(n)}|_{z=0}=(-1)^n(n+1)!(a^n+b^n)$, for any $n=0,1\dots$. This derivative can be computed also by Cauchy formula: $f^{(n)}|_{z=0}=\frac{n!}{2\pi i}\oint\limits_{|\xi|=1}\frac{f(\xi)}{\xi^{n+1}}d\xi$. Therefore:

$(n+1)!|a^n+b^n|=|f^{(n)}|_{z=0}|=\frac{n!}{2\pi}|\oint\limits_{|z|=1}\frac{f(\xi)}{\xi^{n+1}}d\xi|\le 3\frac{n!}{2\pi}\oint\limits_{|z|=1}|d\xi|=3\cdot n!$. Hence the needed inequality.

(3) (This is close to questions 1c, 1.d, 1.h. of hwk8, and question 4.d of hwk 9.)

(a) By the assumption, $f(z)\underset{z\to 0}{\longrightarrow}\infty$ hence f has a pole at $z=0$. Thus we can present $f(z)=\frac{g(z)}{z^n}$, where $n=-ord_{z=0}f$, $g\in\mathcal{O}(\mathbb{C})$ and $g(0)\neq 0$. By the assumption: $|g(z)|\ge|z^{n-\sqrt{2}}|$. Therefore g does not vanish on $\mathbb{C}$.
Note that $n>\sqrt{2}$, therefore g has a pole at ∞. Thus it is necessarily a polynomial (by HW 8 problem 2a). But g has no zeroes on $\mathbb{C}$, thus it must be a constant, contradicting the existence of a pole at ∞.

(Another reasoning: g has no zeros on $\mathbb{C}$, and does not even approach 0, because of the condition $|g(z)|\ge|z^{n-\sqrt{2}}|$. Therefore the image of g is not dense in $\mathbb{C}$.)

(b) By the assumption, f has an isolated singular point at $z=0$. The condition $|f(z)|\ge\frac{1}{|z|\sqrt{2}}$ implies: $\lim\limits_{z\to 0}|f(z)|=\infty$. Thus f is meromorphic on $\bar{\mathbb{C}}$ and we get back to the previous case.

(4) (This is close to question 3.h hwk6, question 4.c of hwk12, and tutorials 6,12.)

By the assumption: $f(z)=u(z)+i\cdot v(z)$ is holomorphic in $\mathbb{C}$ and satisfies: $-Im(f)^2\le Re(f)\le Im(f)^2$. This means: the image of f lies inside the set $\{(x,y)|\ -y^2\le x\le y^2\}$. But then the image cannot be dense in $\mathbb{C}$. Therefore f must be a constant function. And u,v as well.

(5) (Similar to hwk7, q.2.b.)

Solution 1. By the assumption f is holomorphic, thus it is continuous, thus $f(0)=\lim\limits_{z\to 0}f(z)=0$. Let $C=\sup\limits_{z\in\partial D_1(0)}|f(z)|$ and define $\tilde{f}(z)=\frac{f(z)}{C}$. (Note: $C\neq 0$.) Then $D_1(0)\overset{\tilde{f}}{\to}D_1(0)$ satisfies the assumptions of Schwarz lemma. Thus $|\tilde{f}(z)|\le|z|$ for any $z\in D_1(0)$, with equality iff $\tilde{f}(z)=\tilde{C}\cdot z$. The condition on $f(\frac{i}{2})$ means $|\tilde{f}(\frac{i}{2})|=\frac{1}{2}=|\frac{i}{2}|$, therefore $\tilde{f}(z)=\tilde{C}\cdot z$, thus $f(z)=C\tilde{C}\cdot z$. Finally: $\frac{f'(0)}{f(-1)}=-1$.

Solution 2. First notice (as above) that $f(0)=0$. Therefore the function $g(z)=\frac{f(z)}{z}$ is holomorphic in $D_2(0)$. This function satisfies: $|g(\frac{i}{2})|=\sup\limits_{z\in\partial D_1(0)}|g(z)|$. Therefore, by the principle of maximum, g is a constant. Thus $f(z)=Cz$, hence $\frac{f'(0)}{f(-1)}=-1$.

(6) First note that the integral converges absolutely. (The denominator does not vanish for any $t\in\mathbb{R}$ and it decays as $\frac{1}{t^2}$ for $|t|\to\infty$.)

Solution 1. We convert the integral to an integral over the unit circle by the change of variables $t\to z(t)=\frac{i-t}{i+t}$. (Note that $|\frac{i-t}{i+t}|=1$ for any $t\in\mathbb{R}$. And as t varies from $-\infty$ to ∞, z runs counterclockwise along the circle.) We have: $t=i\frac{1-z}{1+z}$ and $dt=\frac{-2idz}{(1+z)^2}$. Thus the integral to compute is:

$$\int\limits_{-\infty}^{\infty}\frac{dt}{(i+t)^2sin(\frac{i-t}{i+t}-\frac{1}{3})}=\lim_{R\to\infty}\int\limits_{-R}^{R}\frac{dt}{(i+t)^2sin(\frac{i-t}{i+t}-\frac{1}{3})}=\lim_{\epsilon\to 0}\oint\limits_{\substack{|z|=1\\ Arg(z)\in[-\pi+\epsilon,\pi-\epsilon]}}\frac{-2idz}{(1+z)^2sin(z-\frac{1}{3})(\frac{2i}{1+z})^2}=$$

$$=\oint\limits_{|z|=1}\frac{-2idz}{(1+z)^2sin(z-\frac{1}{3})(\frac{2i}{1+z})^2}=\oint\limits_{|z|=1}\frac{idz}{2sin(z-\frac{1}{3})}=2\pi i\cdot Res_{z=\frac{1}{3}}\Big(\frac{i}{2sin(z-\frac{1}{3})}\Big)=-\pi$$

Solution 2. Close the contour of integration by the upper semi-circle γ_R (that passes through the points $R, Ri, -R$).

We claim: $\lim\limits_{z\to\infty} \int\limits_{\gamma_R} \frac{dz}{(i+z)^2 sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right)} = 0$. Indeed $\lim\limits_{z\to\infty} sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right) = sin(-\frac{4}{3})$. Therefore for $|z| \gg 1$ one has:

$$|sin\frac{4}{3}| - \epsilon \leq |sin(\frac{i-z}{i+z} - \frac{1}{3})| \leq |sin\frac{4}{3}| + \epsilon.$$

Thus $|\int\limits_{\gamma_R} \frac{dz}{(i+z)^2 sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right)}| \leq C \int\limits_{\gamma_R} |\frac{dz}{(i+z)^2}| \leq \frac{2\pi CR}{R^2}$, for some constant C.

Therefore:

$$(1) \quad \int\limits_{-\infty}^{\infty} \frac{dt}{(i+t)^2 sin\left(\frac{i-t}{i+t}-\frac{1}{3}\right)} = \lim_{R\to\infty}\Big(\int\limits_{-R}^{R} \frac{dt}{(i+t)^2 sin\left(\frac{i-t}{i+t}-\frac{1}{3}\right)} + \int\limits_{\gamma_R} \frac{dz}{(i+z)^2 sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right)}\Big) =$$

$$= 2\pi i \sum_{Im(w)>0} Res_{z=w}\left(\frac{1}{(i+z)^2 sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right)}\right)$$

Thus we should check the singularities/residues of $\frac{1}{(i+z)^2 sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right)}$. Note that $\frac{1}{(i+z)^2}$ is holomorphic for $Im(z) > 0$. So the residues could come only from the points where $sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right) = 0$, i.e. $\frac{i-z}{i+z} - \frac{1}{3} = \pi k$, for $k \in \mathbb{Z}$. We get then: $\{z = -i - \frac{2i}{3\pi k+4}\}_{k\in\mathbb{Z}}$. Of this infinite sequence of points, only one point lies in the upper half plane: $z = \frac{i}{2}$. At this point the function has just a simple pole, therefore:

$$\int\limits_{-\infty}^{\infty} \frac{dt}{(i+t)^2 sin\left(\frac{i-t}{i+t}-\frac{1}{3}\right)} = 2\pi i \cdot Res_{z=\frac{i}{2}}\left(\frac{1}{(i+z)^2 sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right)}\right) = 2\pi i \cdot \lim_{z\to\frac{i}{2}} \frac{z-\frac{i}{2}}{(i+z)^2 sin\left(\frac{i-z}{i+z}-\frac{1}{3}\right)} = -\pi.$$