Partial solutions of moed.B, Complex.Functions.EE

(201.1.0071) 10.08.2017 Ben Gurion University

(1) Note that $|f(z)| = \sqrt{Re(f(z))^2 + Im(f(z))^2}$, therefore the condition $|f(z)| \leq |Re(f(z))|$ implies: $Im(f(z)) \equiv 0$ for any $z \in \mathbb{C}$. But then f must be constant. (e.g. by Cauchy-Riemann equations, or because the image of non-constant entire function must be dense in $\mathbb{C}$, or by the open mapping theorem.)

(2) If $f^{2016} \equiv 0$ then $f \equiv 0$, in particular f is holomorphic. Assume f^{2016} does not vanish identically, then (by holomorphicity) its zeros are isolated.

The function $\frac{f^{2017}}{f^{2016}}$ is defined (and is holomorphic) outside of the set of zeros of f^{2016}. Near each zero of f^{2016}, i.e. $f^{2016}(z_0) = 0$, the function $\frac{f^{2017}}{f^{2016}}$ has an isolated singular point. This point is of "removable" type, because $\lim_{z \to z_0} \frac{f^{2017}(z)}{f^{2016}(z)} = 0$. Thus the function $\frac{f^{2017}}{f^{2016}}$ extends to a holomorphic function defined on the whole $D_1(0)$, and it is precisely f. Thus $f \in \mathcal{O}(D_1(0))$.

(3) <u>*Solution 1.*</u> Recall the minimum principle (see the lectures and the homeworks):

1. If $g \in \mathcal{O}(\overline{\mathcal{U}})$ does not vanish at any point of $\mathcal{U}$ then $\inf_{z \in \overline{\mathcal{U}}} |g|$ is achieved on the boundary.
2. If this infimum is achieved at some inner point then the function is constant.

In our case the function e^f is holomorphic and does not vanish anywhere in $\mathbb{C}$. For any two numbers $r < R$, the points of the circle $|z| = r$ are the inner points for the disc $D_R(0)$. Thus we have the strict inequality.

<u>*Solution 2.*</u> Note that $\inf |e^{f(z)}| = \frac{1}{\sup |e^{-f(z)}|}$. Therefore, instead of proving $\inf_{|z|=r} |e^{f(z)}| > \inf_{|z|=R} |e^{f(z)}|$, it is enough to prove: $\sup_{|z|=r} |e^{-f(z)}| < \sup_{|z|=R} |e^{-f(z)}|$. But the later follows immediately from the maximum principle for the (non-constant) holomorphic function $e^{-f(z)}$.

<u>*Solution 3.*</u> Present $f = u + iv$, note that $|e^{f(z)}| = e^{u(z)}$. Therefore we should prove: $\inf_{|z|=r} e^{u(z)} > \inf_{|z|=R} e^{u(z)}$. As the (real-valued) function e^x is strictly increasing, it is enough to prove: $\inf_{|z|=r} u(z) > \inf_{|z|=R} u(z)$. But this follows immediately by the minimum principle for the (non-constant) harmonic function $u \in Har(\mathbb{C})$.

Note: The function $e^{u(z)}$ is not necessarily harmonic, and the minimum principle does not hold for it.

(4) (Part b.) We want to use Rouché's theorem.

Note that $|e^z| < 1$ for $Re(z) < 0$, thus $|e^z - 1| < 2$ in the whole $\mathcal{U}$. It remains to bound the right hand side, $|z^3 + 7z^2| = |z^2(z+7)|$.

Note that for any point $z \in \partial\mathcal{U}$ holds: $|z^2| \geq 25$. In addition, $|z+7| \geq 2$, here 2 is the distance from the point (-7) to the boundary $\partial\mathcal{U}$. Thus $|z^3 + 7z^2| \geq 50$.

Therefore $|z^3 + 7z| > |e^z - 1|$ on $\partial\mathcal{U}$. Thus, by Rouché theorem, the equations $z^2(z+7) = 0$ and $z^2(z+7) = e^z - 1$ have the same number of solutions in $\mathcal{U}$. The equation $z^2(z+7) = 0$ has just one solution in $\mathcal{U}$. Thus the equation $z^2(z+7) = e^z - 1$ has precisely one solution in $\mathcal{U}$.

(5) <u>*Solution 1.*</u> Recall the Cauchy formula (for z_0 inside γ): $f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)dz}{(z-z_0)}$. From here one gets, by differentiation: $f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i} \oint_\gamma \frac{f(z)dz}{(z-z_0)^n}$. On the other hand, Cauchy formula for $f^{(n-1)}$ is: $f^{(n-1)}(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f^{(n-1)}(z)dz}{(z-z_0)}$. Thus we get: $\oint_\gamma \frac{f(z)dz}{(z-z_0)^n} = \frac{1}{(n-1)!} \oint_\gamma \frac{f^{(n-1)}(z)dz}{z-z_0}$.

If the point z_0 does not lie inside the path γ, then both parts vanish.

<u>*Solution 2.*</u> We are to prove: $\oint_\gamma \left(\frac{f(z)dz}{(z-z_0)^n} - \frac{1}{(n-1)!} \frac{f^{(n-1)}(z)dz}{z-z_0} \right) = 0$, i.e. $\oint_\gamma \frac{f(z) - \frac{f^{(n-1)}(z)}{(n-1)!}(z-z_0)^{n-1}}{(z-z_0)^n} dz = 0$. Expand the numerator at the point z_0: $f(z) - \frac{f^{(n-1)}(z)}{(n-1)!}(z-z_0)^{n-1} = \sum_{k \geq 0} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k - \frac{1}{(n-1)!} \sum_{k \geq 0} \frac{f^{(k+n-1)}(z_0)(z-z_0)^{k+n-1}}{k!}$.

To compute the integral we need the coefficient of $(z-z_0)^{n-1}$ in this series. And this coefficient vanishes.

Solution 3. Replace the contour γ by a small circle around z_0, and compute the residues.

(6) Note that the integral converges absolutely.

It is natural to close the integration path by a (upper or lower) semi-circle, denote the later by C_R. This reduces the integral to the residues of the function at all its poles.

Note that the poles are at the points $\{z=\pi k+i\}_{k\in\mathbb{Z}}$, and these all lie in the upper half-plane. Thus, if we close the path by the upper semi-circle, we will have to sum an infinite series of residues. Therefore it is simpler to close by the lower semi-circle, $C_R=\{R\cdot e^{-i\cdot\theta},\ \theta\in[0,\pi]\}$, here there are no residues at all.

We claim: $\lim\limits_{R\to\infty}\int\limits_{C_R}\frac{dz}{(z-i)^2 sin(z-i)}=0$. Indeed, $sin(z-i)$ does not approach zero on C_R and one has:

$$|\int\limits_{C_R}\frac{dz}{(z-i)^2 sin(z-i)}|\le C\cdot\int\limits_{C_R}\frac{|dz|}{R^2}\le\frac{\pi C}{R}\underset{R\to\infty}{\longrightarrow}0.$$

Thus, as there are no poles in the lower half-plane, we get: $\int\limits_{-\infty}^{\infty}\frac{dt}{(t-i)^2 sin(t-i)}=0$.