Partial solutions of moed.C, Comlex.Functions.EE

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(1) *Solution 1.* The Cauchy integral presentation formula for $z \in D_4(0)$ reads: $f(z) = \frac{1}{2\pi i} \oint\limits_{|\xi|=8} \frac{f(\xi)\cdot d\xi}{\xi - z}$. From here one gets:

$$|f'(z)| = |\frac{1}{2\pi i} \oint\limits_{|\xi|=8} \frac{f(\xi)\cdot d\xi}{(\xi - z)^2}| \le \frac{c}{2\pi} \oint\limits_{|\xi|=8} \left|\frac{d\xi}{(\xi - z)^2}\right| \overset{\substack{|z|<4,\\ |\xi|=8}}{\le} \frac{c}{2\pi} \cdot \frac{2\pi \cdot 8}{4^2} = \frac{c}{2}.$$

Solution 2. Take any point $z \in D_4(0)$, then the disc $D_4(z)$ lies inside the disc $D_8(0)$. Therefore:

$$|f'(z)| = |\frac{1}{2\pi i} \oint\limits_{|\xi - z|=4} \frac{f(\xi)\cdot d\xi}{(\xi - z)^2}| \le \frac{1}{2\pi} \underset{\xi \in \partial D_4(z)}{max} |f(\xi)| \cdot \frac{2\pi \cdot 4}{4^2} \le \underset{\xi \in \partial D_8(z)}{max} |f(\xi)| \cdot \frac{4}{4^2} = \frac{c}{4} \le \frac{c}{4}.$$

(2) (This is question 7 from homework 8)

Solution 1. One has: $|f(z)| = \sqrt{Re(f)^2 + Im(f)^2}$. As f is holomorphic, the function $|f(z)|^2 = Re(f)^2 + Im(f)^2$ is $\mathbb{R}$-differentiable everywhere. But the function $(e^{|z|})^2 = e^{2|z|} = e^{2\sqrt{x^2+y^2}}$ is not differentiable at $(0,0)$.

Solution 2. Suppose the condition $|f(z)| = e^{|z|}$ is satisfied for any $z \in D_1(0)$. Then the function $\frac{1}{f}$ is holomorphic on $D_1(0)$ and $|\frac{1}{f}| = e^{-|z|} \le 1$. Thus the maximum of $|\frac{1}{f}|$ is achieved at $z = 0$. Thus, by the maximum principle, $\frac{1}{f}$ is constant. Which contradicts $|f(z)| = e^{|z|}$.

Solution 3. Suppose the condition $|f(z)| = e^{|z|}$ is satisfied for any $z \in D_1(0)$. Then the holomorphic function f does not vanish on $D_1(0)$. Therefore (by the minimum and maximum principles) both the minimum and the maximum of $|f|$ are achieved on the boundary $\partial D_1(0)$. But $|f|$ is constant on $\partial D_1(0)$. Thus $|f|$ must be constant on the whole $D_1(0)$. But $|f(z)| = e^{|z|}$, which is not constant.

(3) (This question is similar to question 2 of the midterm.)
The function f is holomorphic inside the strip $\frac{\pi}{2} < |z| < \pi$. Therefore, by Morerra theorem, to establish the (non-)existence of the primitive function, it is enough to check the (non-)vanishing of $\oint\limits_{|z|=r} f(z)dz$ for some $r \in (\frac{\pi}{2}, \pi)$.
This integral equals the sum of residues at the singular points of f in $D_{\frac{\pi}{2}}(0)$.

The singular points are: $z = 0$ and $z = \pm\frac{\pi}{2}$.

• The residue at $z = 0$. Note that $Res_{z=0}(e^{\frac{2}{z}} sin(\frac{1}{z^2})) = 0$, e.g. by looking at the Laurent expansion. (Or by expressing this in terms of Res_∞.) Note that $z = 0$ is a removable singularity for $tan(z) \cdot ctan(\frac{z}{\sqrt{2}})$. Therefore $Res_{z=0}(f) = 0$.

• The residue at $z = \frac{\pi}{2}$. The part $e^{\frac{2}{z}} sin(\frac{1}{z^2})$ is regular, while the part $tan(z) \cdot ctan(\frac{z}{\sqrt{2}})$ has a simple pole. Therefore $Res_{z=\frac{\pi}{2}}(f) = (-1) \cdot ctan(\frac{\pi}{2\sqrt{2}})$.

• The residue at $z = -\frac{\pi}{2}$ is computed similarly, $Res_{z=-\frac{\pi}{2}}(f) = (-1) \cdot ctan(-\frac{\pi}{2\sqrt{2}})$.

Altogether, for $r \in (\frac{\pi}{2}, \pi)$ holds: $\oint\limits_{|z|=r} f(z)dz = 0$. Therefore the primitive function does exist.

(4) (a) The uniqueness theorem for harmonic functions reads: Let $\mathcal{U} \subset \mathbb{C}$ be connected, bounded domain. Let $u_1, u_2 \in Har(\mathcal{U})$ and suppose: either $u_1 \equiv u_2$ on some $D_\epsilon(z_0)$ or $u_1, u_2 \in C^0(\overline{\mathcal{U}})$ and $u_1|_{\partial\mathcal{U}} \equiv u_2|_{\partial\mathcal{U}}$. Then $u_1 \equiv u_2$ on $\mathcal{U}$.
(b) The condition $u(x, 0) = sin(x)$ indicates the presence of $Re(sin(z))$. Thus we write:

$$u(x, y) = Re(sin(x + iy)) + v(x + iy).$$

Here v must be harmonic and must satisfy: $v(x, 0) = 0$, $v(0, 1) = \sqrt{2}$. The natural candidate is $v(x, y) = \sqrt{2} \cdot y$. Altogether: the function $u(x, y) = Re(sin(x + iy)) + \sqrt{2} \cdot y = cosh(y) \cdot sin(x) + \sqrt{2} \cdot y$ is harmonic and satisfies the prescribed conditions.

Remark: this function is non-unique, e.g. $e^{\pm y} sin(x) + \sqrt{2} \cdot y$ also satisfies the conditions.

(5) By the assumption, the integrand $\frac{f(\xi)}{(\xi-2)(\xi+2)(\xi-2i)}$ is holomorphic in $D_1(0)$. Therefore, by Cauchy theorem, $F(z)$ does not depend on the choice of the integration path. Moreover, F is holomorphic in $D_1(0)$. Therefore $\oint\limits_{|z|=\frac{1}{2}} F(z)dz = 0$.

(6) We choose the orientation of the path: from $\frac{1}{2} + i \cdot \infty$ to $\frac{1}{2} - i \cdot \infty$. First we observe that the integral converges (absolutely). We want to close the integration path to reduce the computation to residues.

For this we check the singularities of the integrand. The expression $\frac{1}{(e^{z+1}-1)(z^3-1)}$ has poles at $\{z^3 = 1\}$ and at $\{z + 1 \in 2\pi i \cdot k\}_{k\in\mathbb{Z}}$. Of these points, the point $z = 1$ is to the right of the curve, while the others are to the left of the curve. To avoid working with the infinite number of poles, we close the path by a curve lying in the right half-plane $\{Re(z) > \frac{1}{2}\}$. For example, by a part of semi-circle, denote it by C_R.

We claim: $\lim\limits_{R\to\infty} \int\limits_{C_R} \frac{dz}{(e^{z+1}-1)(z^3-1)} = 0$. (For example, because for large R holds: $\left|\frac{1}{(e^{z+1}-1)(z^3-1)}\right| < \frac{10}{R^3}$.) Therefore:

$$\int\limits_{\gamma} \frac{dz}{(e^{z+1}-1)(z^3-1)} = \int\limits_{\gamma} \frac{dz}{(e^{z+1}-1)(z^3-1)} + \lim_{R\to\infty} \int\limits_{C_R} \frac{dz}{(e^{z+1}-1)(z^3-1)} =$$

$$= 2\pi i \sum Res \left(\frac{1}{(e^{z+1}-1)(z^3-1)} \right) = 2\pi i Res_{z=1} \left(\frac{1}{(e^{z+1}-1)(z^3-1)} \right) = \frac{2\pi i}{3(e^2-1)}.$$

Note: if the orientation of the path is taken from $\frac{1}{2} - i\cdot\infty$ to $\frac{1}{2} + i\cdot\infty$, then one has:

$$\int\limits_{\gamma} \frac{dz}{(e^{z+1}-1)(z^3-1)} = -2\pi i \sum Res_{z=z_j} \left(\frac{1}{(e^{z+1}-1)(z^3-1)} \right) = -\frac{2\pi i}{3(e^2-1)}.$$