Solutions of Moed.A in Hedva2.ME

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(1) First we look for the critical points inside the ball $\{x^2+y^2+z^2<1\} \subset \mathbb{R}^3$. The condition $grad(f)=(0,0,0)$ gives only one point: $(x,y,z)=(0,0,1)$. This point is on the boundary of the ball and will be addressed lated.

Now we consider the function on the sphere $\{g(x,y,z)=x^2+y^2+z^2-1=0\} \subset \mathbb{R}^3$. We can use Lagrange's method, either in the form of the function $F(x,y,z,\lambda)$ or using the condition: $grad(f) \sim grad(g)$.

The approach using Lagrange's function. We define $F(x,y,z,\lambda)=f(x,y,z)-\lambda\cdot g(x,y,z)$ and look for the critical points of F. Then $grad(F)=\vec{0}$ gives the system of equations:

$$2x+y=\lambda\cdot 2x, \quad x+2y=\lambda\cdot 2y, \quad 2z-2=\lambda\cdot 2z.$$

From the first two equations we get: $3(x+y)=2\lambda(x+y)$ and $x-y=2\lambda(x-y)$. Which means: $(x+y)(3-2\lambda)=0$ and $(x-y)(1-2\lambda)=0$. We get: either $\Big(x=0=y\Big)$ or $\Big(x=y\neq 0 \text{ and } \lambda=\frac{3}{2}\Big)$ or $\Big(x=-y\neq 0 \text{ and } \lambda=\frac{1}{2}\Big)$.

For $\lambda=\frac{3}{2}$ we get $z=-2$, for $\lambda=\frac{1}{2}$ we get: $z=2$. In both cases the point cannot lie on the sphere $\{x^2+y^2+z^2=1\}$. The remaining case, $x=0=y$, gives: $z=\pm 1$.

The approach via $grad(f)\sim grad(g)$. This condition is: $rank\begin{pmatrix}2x+y & x+2y & 2z-2\\ 2x & 2y & 2z\end{pmatrix}<2$. By row operations on this matrix the condition simplifies to: $rank\begin{pmatrix}y & x & -2\\ x & y & z\end{pmatrix}<2$. This condition is translated into equations by writing down the maximal minors:

$$y^2=x^2, \quad yz+2x=0, \quad xz+2y=0.$$

This implies: either $x=y=0$ or $\Big(x=y\neq 0 \text{ and } z=-2\Big)$ or $\Big(x=-y\neq 0 \text{ and } z=2\Big)$.

The points with $z=\pm 2$ do not lie on the sphere. Thus the only points to check are: $x=0=y$, $z=\pm 1$.

In either way, once we find the points, we get: $f(0,0,-1)=3=max$ and $f(0,0,1)=-1=min$.

(2) We want to pass from the triple integral to a repeated integral. By the direct check, the suitable order of integration is $\int(\int(\int dy)dx)dz$. One has:

$$\iiint\limits_V \cdots = \int_0^1 \Big(\int_0^{\sqrt{z}} \big(\int_0^{\sqrt{x}} |y| e^{x^2} dy\big)dx\Big)dz \overset{y\geq 0}{=} \int_0^1 \Big(\int_0^{\sqrt{z}} e^{x^2}\big(\int_0^{\sqrt{x}} y dy\big)dx\Big)dz = \int_0^1\Big(\int_0^{\sqrt{z}} e^{x^2}\frac{x}{2}dx\Big)dz = \int_0^1\Big(\frac{e^z-1}{4}\Big)dz = \frac{e-2}{4}.$$

Remark: this integral is also computable in the following order: $\int\limits_0^1\Big(\int\limits_0^{\sqrt{x}}\big(\int\limits_{x^2}^1 |y|e^{x^2}dz\big)dy\Big)dx$.

(3) The integral is over an 'unpleasant' curve $\gamma=\{x^{10}+y^{10}=100,\ x\leq 0\}$. This curve is not closed. We close it by the (simpler) curve $C=\big\{x^2+y^2=100^{\frac{2}{10}},\ x\leq 0\big\}$ and use Green's theorem. The orientation of C is taken counterclockwise. Denote by $\mathcal{D}$ the domain whose boundary is the union $\gamma\cup C$. Note that $\mathcal{D}$ does not contain the origin, thus the field of integration is well behaved on $\mathcal{D}$ (has continuous partial derivatives). We get:

$$\int\limits_\gamma \frac{ydx-xdy}{x^2+y^2} - \int\limits_C \frac{ydx-xdy}{x^2+y^2} = \iint\limits_{\mathcal{D}} \Big(\partial_x\frac{-x}{x^2+y^2} - \partial_y\frac{y}{x^2+y^2}\Big)dxdy = 0$$

Therefore it is enough to compute $\int\limits_C \frac{ydx-xdy}{x^2+y^2}$, e.g. by parametrization, $\{\big(x(t),y(t)\big)=\big(100^{\frac{1}{10}}cos(t), 100^{\frac{1}{10}}sin(t)\big),\ t\in[\frac{\pi}{2},\frac{3\pi}{2}]\}$. We get:

$$\int\limits_{\substack{x^2+y^2=100^{\frac{2}{10}},\\ x\leq 0}} \frac{ydx-xdy}{x^2+y^2} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\big(-cos^2(t)-sin^2(t)\big)dt = -\pi.$$

Therefore $\int\limits_\gamma \frac{ydx-xdy}{x^2+y^2} = -\pi$.

(4) The surface is naturally parameterized by x, y. These parameters vary in the region $\{x^2 + y^2 + 2y \leq 2017\}$, which is a shifted disc: $\{x^2 + (y + 1)^2 \leq 2018\}$. Therefore, for any function h holds:

$$\iint h \cdot dS = \iint h \cdot \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dxdy = \iint h\sqrt{2}dxdy.$$

Thus

$$Area = \iint\limits_S 1dS = \iint\limits_{\{x^2+y^2+2y\leq 2017\}} \sqrt{2} \cdot dxdy = \sqrt{2} \cdot \Big(\text{area of the disc of radius } \sqrt{2018}\Big) = \sqrt{2} \cdot \pi \cdot (\sqrt{2018})^2.$$

(5) The surface is the boundary of the ellipsoid $V = \{x^2 + 4y^2 + 9z^2 \leq 1\}$.

<u>*Solution 1.*</u> The vector field has continuous partial derivatives in this ellipsoid. Therefore we can use Gauss theorem (note that the prescribed normal is inside, thus we insert the minus sign):

$$\iint\limits_S \vec{F} \cdot d\vec{S} = -\iiint\limits_V div(\vec{F})dV = -\iiint\limits_V (4x^3 + 8y^7 + 1)dV \overset{symmetry}{=\!=} -0 - 0 - \iiint\limits_V 1 \cdot dV = -\begin{pmatrix} volume\ of \\ ellipsoid\ with \\ axes\ (1, \frac{1}{2}, \frac{1}{3}) \end{pmatrix} = -\frac{4\pi}{3}\pi \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{3}.$$

<u>*Solution 2.*</u> Split the ellipsoid into the upper and lower parts, each of them is naturally parameterized by (x, y) coordinates. For each part take the corresponding (inner) normal, $\vec{\mathcal{N}} = -\partial_x \vec{r} \times \partial_y \vec{r}$. Then

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_{\substack{(x,y)\in\mathcal{D} \\ z_+ = \frac{\sqrt{1-x^2-4y^2}}{3}}} \vec{F} \cdot \vec{\mathcal{N}} dxdy + \iint\limits_{\substack{(x,y)\in\mathcal{D} \\ z_- = -\frac{\sqrt{1-x^2-4y^2}}{3}}} \vec{F} \cdot \vec{\mathcal{N}} dxdy, \quad \mathcal{D} = \{x^2 + 4y^2 \leq 1\}$$

The first two components of the field, x^4, y^8, are even, thus their contributions on the upper and lower parts cancel. Thus, of the whole $\vec{F}$, only the part $(0, 0, z)$ is relevant. Now we compute: $\vec{F} \cdot \vec{\mathcal{N}} = -det \begin{pmatrix} 0 & 0 & z \\ 1 & 0 & \partial_x z \\ 0 & 1 & \partial_y z \end{pmatrix} = -z.$

Therefore the integral is:

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_{(x,y)\in\mathcal{D}} \big(-z_+ + z_-\big)dxdy = -\Big(\text{Volume of the ellipsoid}\Big) = -\frac{4\pi}{3}\pi \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{3}.$$

(6) <u>*Solution 1.*</u> Think of the curve C as the (oriented) boundary of the surface $S = \{x^2 + y^2 + z^2 \leq 1,\ y + z = -1\}$. This surface is a disc. The (prescribed) orientation of the curve determines the direction of the normal to the disc: $\mathcal{N} = (0, 1, 1)$. (Note: the choice $(0, -1, -1)$ is wrong.) The field has continuous partial derivatives on S, therefore we can use Stokes theorem. Note that $rot(\vec{F}) = (1, 1, 1)$ thus we have:

$$\oint\limits_C \vec{F} \cdot d\vec{r} = \iint\limits_S (1, 1, 1) \cdot d\vec{S} = \iint\limits_S (1, 1, 1) \cdot \frac{(0, 1, 1)}{|(0, 1, 1)|} dS = \sqrt{2} \iint\limits_S 1 \cdot dS = \sqrt{2} \cdot \big(\text{area of } S\big).$$

The area of S can be computed in various ways. For example, its diameter is the distance between the points $(0, 0, -1)$, $(0, -1, 0)$, hence the radius equals $\frac{1}{\sqrt{2}}$. Altogether we get:

$$\oint\limits_C \vec{F} \cdot d\vec{r} = \sqrt{2} \cdot \pi \cdot \frac{1}{2}.$$

<u>*Solution 2.*</u> We use the following parametrization of the curve: $\gamma(t) = (x(t), y(t), z(t)) = \left(\frac{cos(t)}{\sqrt{2}}, \frac{sin(t)-1}{2}, -\frac{sin(t)+1}{2}\right)$.

Note that $\gamma'(t) = \left(-\frac{\sin(t)}{\sqrt{2}}, \frac{\cos(t)}{2}, -\frac{\cos(t)}{2}\right)$ and $F(\gamma(t)) = \left(-\sin(t) - 1, \frac{\cos(t)}{\sqrt{2}}, \frac{\cos(t)}{\sqrt{2}} + \frac{\sin(t)-1}{2}\right)$.

Therefore the scalar product of these two vectors is: $F(\gamma(t)) \cdot \gamma'(t) = \frac{\sin^2(t)}{\sqrt{2}} + \frac{\sin(t)}{\sqrt{2}} - \frac{\sin(t)\cos(t)}{4} + \frac{\cos(t)}{4}$.

Substitute this into the integral: $\oint\limits_C \vec{F} \cdot d\vec{r} = \int\limits_{t=0}^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt = \ldots$.

The integral over $\sin(t)$ vanishes, and so do the integrals over $\cos(t)$ and $\sin(t)\cos(t) = \frac{\sin(2t)}{2}$. We're finally left with

$$\oint\limits_C \vec{F} \cdot d\vec{r} = \ldots = \int_{t=0}^{2\pi} \frac{\sin^2(t)}{\sqrt{2}}\, dt = \frac{1}{2\sqrt{2}} \int_{t=0}^{2\pi} 1 - \cos(2t)\, dt = \frac{\pi}{\sqrt{2}}.$$