Solutions of Moed.B in Hedva2.ME

(201.1.9721) 30.07.2017 Ben Gurion University

(1) The domain $\{\frac{2^2}{2} - x + \frac{y^2}{2} + y \leq 0\}$ can be presented in the form $\{(x-1)^2 + (y+1)^2 \leq 2\}$, it is a disc of radius $\sqrt{2}$, centered at $(1, -1)$.

<u>Solution 1.</u> The function $f(x,y) = |xy|$ satisfies: $f(x,y) = f(-x,y) = f(x,-y) = f(-x,-y)$. Therefore, while searching for the critical points in the interior points, we can assume $x \geq 0$ and $y \geq 0$. The only critical point of $xy$ is $(0,0)$. Note that it lies on the boundary of the disc. To this we must add also the points where $f$ is non-differentiable, these are all the points satisfying $xy = 0$. At all such points $f$ vanishes.

When looking for the critical points of $f$ on the boundary of the disc, we again use $f(x,y) = f(-x,y) = f(x,-y) = f(-x,-y)$. Therefore it is enough to check the critical points of $xy$ on the circle $(x-1)^2 + (y+1)^2 = 2$, and also to check the points where the function might have the differentiability problem, i.e. $|xy| = 0$. The critical points of $xy$ on the circle $(x-1)^2 + (y+1)^2 = 2$ are obtained in the standard way. (By Lagrange multiplier's method, or by the condition $grad(xy) \sim grad((x-1)^2 + (y+1)^2 - 2)$.) One gets the condition $(y - x + 1)(y + x) = 0$. Together with $(x-1)^2 + (y+1)^2 = 2$ one gets:

- Either $x = -y = 0$, with $f(0,0) = 0$;
- Or $x = -y = 2$, with $f(-2,2) = 4$;
- Or $y + 1 = x$, with $x = \frac{1 \pm \sqrt{3}}{2}$. Here: $f(\frac{1\pm\sqrt{3}}{2}, \frac{-1\pm\sqrt{3}}{2}) = |\frac{-1-3}{2}| = 2$.

Thus the minimal value of $f$ is 0, while the maximal is 4.

<u>Solution 2.</u> As $f(x,y) = |xy|$, one has: $f \geq 0$. Thus the minimal value of $f$ is 0 and it is obtained at the set of points where $xy = 0$. To find the maximal value of $f$ one can pass to the polar coordinates, then the expression for $f$ is: $r^2|\frac{sin(2\phi)}{2}|$. Now note that $|sin(2\phi)| \leq 1$ and attains 1 for $\phi = -\frac{\pi}{4}$. In addition, by drawing the circle $\{(x-1)^2 + (y+1)^2 = 2\}$, one gets: the maximal value of $r$ is $2\sqrt{2}$ and is attained for $\phi = -\frac{\pi}{4}$. Altogether we get: $f(x,y) \leq 4$ and this value is achieved at the point $(2, -2)$.

(2) We pass from the triple integral to the repeated integral, projecting onto the $yz$-plane. The projection is the domain bounded by two ellipses. Thus we have:

$$(1) \quad \iiint\limits_V \frac{x}{1+x^2} dxdydz = \iint\limits_{\{\frac{1}{2} \leq 4y^2+9z^2 \leq 1\}} \Big( \int\limits_0^{\sqrt{1-4y^2-9z^2}} \frac{x}{1+x^2} dx \Big) dydz = \iint\limits_{\{\frac{1}{2} \leq 4y^2+9z^2 \leq 1\}} \frac{ln(2-4y^2-9z^2)}{2} dydz \overset{\tilde{y}=2y}{\underset{\tilde{z}=3z}{=}}$$

$$= \iint\limits_{\{\frac{1}{2} \leq \tilde{y}^2+\tilde{z}^2 \leq 1\}} \frac{ln(2-\tilde{y}^2-\tilde{z}^2)}{2 \cdot 2 \cdot 3} d\tilde{y}d\tilde{z} = 2\pi \int\limits_{\{\frac{1}{\sqrt{2}} \leq r \leq 1\}} \frac{ln(2-r^2)}{2 \cdot 2 \cdot 3} r \cdot dr \overset{t=r^2}{=} \frac{2\pi}{24} \int\limits_{\frac{1}{2}}^{1} ln(2-t)dt = \frac{2\pi}{24} \left( \frac{3}{2} ln(\frac{3}{2}) + \frac{1}{2} \right).$$

(3) <u>Solution 1.</u> The domain is $\mathcal{D} = \{x^{\frac{2}{3}} + y^2 \leq 1\}$. Its area, $\iint\limits_{\mathcal{D}} 1dS$, can be computed in different ways. For example, introduce the appropriate modification of polar coordinates: $x = r^3 \cdot cos^3(\phi)$, $y = r \cdot sin(\phi)$. In these coordinates: $\mathcal{D} = \{\phi \in [0, 2\pi],\ r \in [0,1]\}$. The Jacobian of this coordinate change is: $det|\frac{\partial(x,y)}{\partial(r,\phi)}| = 3r^3 cos^2(\phi)$. Finally:

$$\iint\limits_{\mathcal{D}} 1dS = \iint\limits_{\substack{\phi \in [0,2\pi], \\ r \in [0,1]}} 3r^3 cos^2(\phi) drd\phi = \frac{3}{4} \int\limits_0^{2\pi} \frac{1 + cos(2\phi)}{2} d\phi = \frac{3}{8} 2\pi.$$

<u>Solution 2.</u> Recall that the area can be computed using Green's formula for a particular vector field: $\oint\limits_{\partial S} xdy = \iint\limits_S 1dS$. Using this we have:

$$\iint\limits_S 1dS = \oint\limits_{\partial S} xdy = \int\limits_0^{2\pi} cos^4(t)dt = \frac{3}{4}\pi.$$

(4) The surface is naturally parameterized by $x, y$, their domain of variation is: $\mathcal{D} = \{x^2 + y^2 \leq 2,\ x + y \geq 0\}$. Therefore the needed area equals:

$$\iint\limits_S 1dS = \iint\limits_{\mathcal{D}} \sqrt{1 + (\partial_x z)^2 + (\partial_y z)^2} dxdy = \iint\limits_{\substack{x^2+y^2 \leq 2, \\ x+y \geq 0}} \sqrt{1 + 4(x^2+y^2)} dxdy = \iint\limits_{\substack{0 \leq r \leq \sqrt{2} \\ \phi \in [-\frac{\pi}{4}, \frac{3\pi}{4}]}} \sqrt{1+4r^2} d\phi(r \cdot dr) = \frac{13\pi}{6}.$$

(5) Note that $div(\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}})=0$. Therefore we can use Gauss theorem to replace the surface by a simpler one, as far as the field remains differentiable in the whole body of integration.

*Solution 1.* First we note: $\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\leq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, as both the field and the normal are anti-symmetric under $(x,y,z)\leftrightarrow(-x,-y,-z)$. Therefore

$$\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S} = \frac{1}{2}\iint\limits_{|x|^3+|y|^3+|z|^3=1} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S}.$$

To compute this later integral we would like to use Gauss theorem, but the field is not differentiable at $(0,0,0)$. Thus we use Gauss theorem for the body: $\{|x|^3+|y|^3+|z|^3\leq 1,\ x^2+y^2+z^2\geq\epsilon^2\}$. Here $0<\epsilon<1$ is a (small) constant. As $div(\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}})=0$ we have:

$$\frac{1}{2}\iint\limits_{|x|^3+|y|^3+|z|^3=1} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S} = -\frac{1}{2}\iint\limits_{x^2+y^2+z^2=\epsilon^2} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S}, \quad \text{with the inner normal.}$$

For the later integral we note that $x^2+y^2+z^2\equiv\epsilon^2$ along the surface, thus

$$-\frac{1}{2}\iint\limits_{x^2+y^2+z^2=\epsilon^2} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S} = -\frac{1}{2}\iint\limits_{x^2+y^2+z^2=\epsilon^2} \frac{(x,y,z)}{\epsilon^3}\cdot d\vec{S} \overset{Gauss}{=\!=} \frac{1}{2\epsilon^3}\iiint\limits_{x^2+y^2+z^2\leq\epsilon^2} 3dV = \frac{4\pi}{2}.$$

*Solution 2.* We would like to replace the initial surface by the planar domain $\{|x|^3+|y|^3\leq 1,\ z=0\}$. But this domain contains the point $(0,0,0)$, where the field is not differentiable. Thus we replace the initial surface by the union:

$$\underbrace{\left\{\begin{array}{l} |x|^3+|y|^3\leq 1,\ z=0\\ x^2+y^2\geq\epsilon^2\end{array}\right\}}_{S_1}\cup\underbrace{\{x^2+y^2+z^2=\epsilon^2, z\geq 0\}}_{S_2}.$$

As $div(\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}})=0$ we have by Gauss theorem:

$$\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \iint\limits_{S_1} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \iint\limits_{S_2} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = 0.$$

In the later two integrals the normals are taken downstairs.

The normal to $S_1$ is $(0,0,-1)$, therefore for the integral over $S_1$ we have: $(x,y,z)\cdot d\vec{S}=-zdS=0$, as $z=0$. Thus

$$\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \iint\limits_{S_2} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \text{with the normal upstairs.}$$

The later integral is computed in various ways, e.g.

$$\iint\limits_{S_2} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \iint\limits_{S_2} \frac{\vec{r}\cdot\frac{\vec{r}}{r}dS}{r^3} = \iint\limits_{S_2} \frac{dS}{r^2} = \frac{1}{\epsilon^2}\iint\limits_{S_2} dS = \frac{\text{area of } S_2}{\epsilon^2} = \frac{1}{\epsilon^2}2\pi\epsilon^2.$$

(6) The integration path bounds the disc, $D$, we apply Stokes theorem: $\oint\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_D (1,-1,1)\cdot d\vec{S}$. By the assumption on the direction of the path, the (unit) normal to the disc is: $\frac{(0,1,-1)}{\sqrt{2}}$. Therefore:

$$\oint\limits_C \vec{F}\cdot d\vec{r} = -\sqrt{2}\cdot(\text{the area of } D).$$

The diameter of $D$ is obtained as the distance between the points $(0,0,0)$ and $(0,1,1)$, which is $\sqrt{2}$.

Therefore $\oint\limits_C \vec{F}\cdot d\vec{r} = -\sqrt{2}\pi(\frac{1}{\sqrt{2}})^2$.