Solutions of Moed.C in Hedva2.ME

(201.1.9721) 06.09.2017 Ben Gurion University

(1) Note that $ln(t)$ is an increasing function. Therefore it is enough to find the global minimum/maximum of the function $f(x,y) = x^2 + y^2$ on the curve. The condition for the critical point is: $grad(f) \sim grad(g)$. This means: $rank \begin{pmatrix} 2x & 2y \\ 3x-y & 3y-x \end{pmatrix} < 2$. Thus the condition is: $x^2 = y^2$. We get:

- either $x = y$ and then $x + y = \pm 2$. So the points are $(1,1)$ and $(-1,-1)$.
- or $x = -y$ and then $x - y = \pm 1$. So the points are $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$.

Thus the global maximum of $ln(f)$ is achieved at the points $(1,1)$, $(-1,-1)$, and equals $ln(2)$. The global minimum is achieved at the points $(\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, and equals $-ln(2)$.

(2) The integration in the given order is unpleasant, therefore we first return to the triple integral, $\iiint\limits_V \frac{|z| dxdydz}{x^4+1}$, where

$$V = \{-1 \le z \le 0, 0 \le y \le 8, \ \sqrt[3]{y} \le x \le 2\} = \{-1 \le z \le 0, 0 \le y \le x^3, \ 0 \le x \le 2\}.$$

Therefore: $\iiint\limits_V \frac{|z| dxdydz}{x^4+1} = \int\limits_0^2 dx \int\limits_0^{x^3} dy \int\limits_{-1}^0 \frac{|z| dz}{x^4+1} = \frac{1}{2} \int\limits_0^2 \frac{x^3 dx}{x^4+1} = \frac{ln(17)}{8}$.

(3) Rewrite the integral in the form $\oint\limits_C \left(\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \right) + \oint\limits_C ydx$. The part $\oint\limits_C ydx$ is computed immediately using the parametrization $C = \{(cos(\phi), \frac{sin(\phi)}{\sqrt{2}}), \ \phi \in [0, 2\pi]\}$. We have (note that C is oriented counterclockwise):

$$\oint\limits_C ydx = -\int\limits_0^{2\pi} (-\frac{sin^2(\phi)}{\sqrt{2}}) d\phi = \frac{\pi}{\sqrt{2}}.$$

For the first part we use Green's theorem in the domain $\mathcal{D} = \{x^2 + 2y^2 \le 1, \ x^2 + y^2 \ge \epsilon^2\}$. In this domain the vector field is differentiable and we get:

$$\oint\limits_C \left(\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \right) - \oint\limits_{\substack{x^2+y^2=\epsilon^2 \\ clockwise}} \left(\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \right) = -\iint\limits_{\mathcal{D}} \left(-\partial_x \frac{x}{x^2+y^2} - \partial_y \frac{ydx}{x^2+y^2} \right) = 0.$$

(Here the minus sign in front of $\iint\limits_{\mathcal{D}}$ is because both curves are clockwise oriented.) Thus we have:

$$\oint\limits_C \left(\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \right) = -\oint\limits_{\substack{x^2+y^2=\epsilon^2 \\ counterclockwise}} \left(\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \right) \overset{\substack{x = cos(\phi) \\ y = sin(\phi)}}{=\!=} -\int\limits_0^{2\pi} (-1) d\phi = 2\pi.$$

Altogether: $\oint\limits_C \left(\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \right) + \oint\limits_C ydx = 2\pi + \frac{\pi}{\sqrt{2}}$.

(4) The surface $\{y^2 = x^2 + z^2 + 1\}$ is a hyperboloid with two parts, one lies in the region $y \ge 1$, the other lies in the region $y \le -1$. Therefore the prescribed integral is: $\iint\limits_{\substack{y^2=x^2+z^2+1 \\ 1 \le y \le 2}} ydS$. This surface is the graph of function, $y = \sqrt{x^2+z^2+1}$, over the domain $\{x^2 + z^2 \le \sqrt{3}\}$. Therefore

$$\iint\limits_{\substack{y^2=x^2+z^2+1 \\ 1 \le y \le 2}} ydS = \iint\limits_{x^2+z^2 \le \sqrt{3}} \sqrt{x^2+z^2+1} \cdot \sqrt{1 + (\partial_x y)^2 + (\partial_z y)^2} \cdot dxdz = \iint\limits_{x^2+z^2 \le \sqrt{3}} \sqrt{x^2+z^2+1} \cdot \sqrt{\frac{1+2x^2+2y^2}{1+x^2+z^2}} \cdot dxdz =$$

$$= \int\limits_0^{2\pi} \int\limits_0^{\sqrt{3}} \sqrt{1+2r^2} \cdot rdrd\phi = \pi \int\limits_0^3 \sqrt{1+2t} \cdot dt = \frac{14\pi}{3}.$$

(5) To use Gauss theorem we close the surface by the cap $S_1 = \{z = 0, \ x^2 + y^2 \le 1\}$, with the normal downstairs. Then $S \cup S_1$ is a closed surface with the normal inside. Therefore Gauss theorem gives:

$$\iint\limits_S (y^{2017}, x^{2017}, z) \cdot d\vec{S} + \iint\limits_{S_1} (y^{2017}, x^{2017}, z) \cdot d\vec{S} = -\iint\limits_{\substack{z \le 0 \\ x^2+y^2+2017z^2 \le 1}} div(y^{2017}, x^{2017}, z) \cdot dxdydz =$$

$$= -\iint\limits_{\substack{z \le 0 \\ x^2+y^2+2017z^2 \le 1}} dxdydz = -\iint\limits_{\substack{z \le 0 \\ x^2+y^2+\tilde{z}^2 \le 1}} dxdy \frac{d\tilde{z}}{\sqrt{2017}} = -\frac{2\pi}{3\sqrt{2017}}.$$

The normal to S_1 is $(0,0,-1)$, therefore $\iint\limits_{S_1} (y^{2017}, x^{2017}, z) \cdot d\vec{S} = -\iint\limits_{x^2+y^2 \le 1} z(x,y) \cdot dxdy = 0$.

Thus $\iint\limits_S (y^{2017}, x^{2017}, z) \cdot d\vec{S} = -\frac{2\pi}{3\sqrt{2017}}$.

(6) The intersection of the surfaces $x^2 + y^2 + z^2 = 3$, $x^2 + y^2 - z = 1$ is a circle that lies in the plane $z = 1$. We think of this circle as the boundary of the disc $\mathcal{D} = \{z = 1,\ x^2 + y^2 \leq 2\}$ and use Stokes theorem. The orientation of the circle corresponds to the normal $(0, 0, -1)$ to $\mathcal{D}$. Therefore

$$\int\limits_C \vec{F} \cdot d\vec{r} = \iint\limits_{\mathcal{D}} rot(\vec{F}) d\vec{S} = \iint\limits_{\substack{z=1 \\ x^2+y^2 \leq 2}} (-2z^2)(-1) dx dy = 4\pi.$$