Partial solutions of Moed.B, Hedva.3.EE

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(1) (a) The conditions $f_x(0,0) = 0 = f_y(0,0)$ do not imply even the continuity of the function, let alone differentiability. The function: $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ satisfies the assumptions, but most directional derivatives do not exist.

Even if all the directional derivatives exist, the function is not necessarily differentiable, and some derivatives may not vanish. For example,

$$f(x,y) = \begin{cases} \frac{y^2}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(b) The assumption $det(H_f(a)) < 0$ implies: either one eigenvalue is negative (and two are positive) or all the eigenvalues are negative.

The later contradicts the assumption that the sum of eigenvalues is zero. Therefore there are eigenvalues of different signs, hence the point is a saddle point. Thus a is not an extremum.

Remark: it follows from the condition $\det(H_f(a)) < 0$ that a cannot be a local minimum. But we cannot conclude from this condition alone that a is a saddle point.

(c) Note that $\vec{F} = grad\frac{ln(x^2+4y^2+z^2)}{2}$ and S is precisely the level surface for the function $ln(x^2+4y^2+z^2)$. Therefore, at each point of S the field $\vec{F}$ is perpendicular to S. Thus $\int\limits_C \vec{F} \cdot d\vec{C} = 0$ for any smooth curve on S.

Alternatively, as $\vec{F} = grad(\phi)$, we get $\int\limits_C \vec{F} \cdot d\vec{C} = \phi(end - point) - \phi(start - point) = 0$

(2) The domain of the definition of f is: $y \neq 0, 1$.

(a) The level curve $f(x,y) = 1$ is the hyperbola $x^2 + \frac{1}{2} = 2(y - \frac{1}{2})^2$, with the punctures at $(0,0)$ and $(\pm\frac{1}{2}, 1)$.

(b) The level curve $f(x,y) = -1$ is the parabola $x^2 = y$, with the punctures at $(0,0)$ and $(\pm 1, 1)$.

(c) The level curve $f(x,y) = -2$ is the circle $x^2 + (y-1)^2 = 1$, with the punctures at $(0,0)$ and $(\pm 1, 1)$.

(3) (a) <u>Solution 1.</u>

The natural parameters for the surface S are (x,y). Therefore the normal is:
$\vec{\mathcal{N}} = -\partial_x\vec{r} \times \partial_y\vec{r} = (2x, 2y, -1)$. (We take here the $-$ sign, to get $\mathcal{N}_z < 0$.) Thus

$$\iint\limits_S \vec{F} d\vec{S} = \iint\limits_{\{x^2+y^2 \leq 1\}} \vec{F} \cdot (2x, 2y, -1) dxdy = \iint\limits_{\{x^2+y^2 \leq 1\}} 2(x^4+y^4) dxdy =$$

$$= 2\int\limits_{r=0}^{1}\int\limits_{\phi=0}^{2\pi} r^4(cos^4(\phi) + sin^4(\phi)) r dr d\phi = 2 \cdot 2\pi \cdot \frac{1}{6} \cdot \frac{3}{4} = \frac{\pi}{2}.$$

underlineSolution 2. Close the surface by the cap $S_1 = \{z = 1, \ x^2+y^2 \leq 1\}$. Note that $\iint\limits_{S_1} \vec{F} \cdot d\vec{S_1} = 0$, as $F_z = 0$.

Therefore, using Gauss theorem we get:

$$\iint\limits_S \vec{F} d\vec{S} = \iiint\limits_{x^2+y^2 \leq z \leq 1} div(\vec{F}) dxdydz \overset{cylindrical}{=} \int\limits_{r=0}^{1}\int\limits_{\phi=0}^{2\pi}\int\limits_{z=r^2}^{1} 3r^3 dr d\phi dz = \frac{\pi}{2}.$$

Remark: you cannot use Gauss' theorem directly to compute this integral, since the surface is not closed. The above argument works only because the surface integral on the "lid" we added was 0. This, of course, has to be justified.

(b) Note that the field F has no z-component, therefore $\oint\limits_\gamma F \cdot d\gamma$ is a curvilinear integral over the closed curve $\{(\frac{cos(t)}{\sqrt{2}}, \frac{sin(t)}{\sqrt{3}}), \ t \in [0, 2\pi]\} \subset \mathbb{R}^2$.

This curve bounds the domain $\{2x^2 + 3y^2 \leq 1\}$ and the field is differentiable in this domain. Thus we can use Green's formula,

$$\oint\limits_\gamma F \cdot d\gamma = \iint\limits_{\{2x^2+3y^2 \leq 1\}} (0-0) dxdy = 0$$

Alternatively, note that γ is a simple closed curve on the surface S of the previous clause. So we may use Stokes' theorem. Since $curl(F) = 0$ (obviously), we get the same result.

Remark you cannot use Green's theorem directly to compute a line integral in $\mathbb{R}^3$. To use Green's theorem you have to reduce the problem to one in $\mathbb{R}^2$. But this has to be justified, as in the first solution. To use Stokes' theorem you have to provide a (parametric) surface of which γ is the boundary. This has to be justified, as – for example – in Solution 2.

(4) (a) It is enough to prove: the intersection has a C^1-parametrization locally at each of its points.
Solution 1. From the second equation extract $z = 10 - x - y$ and substitute to the first equation. Then it is enough to prove: the equation $f(x, y) = x^3 + y^3 + 3(10 - x - y) = 0$ admits a C^1-solution, $y(x)$ or $x(y)$, at each point. This can be achieved by the implicit function theorem, once we check that $grad(f(x, y)) \neq (0, 0)$. Indeed, the condition $grad(f(x, y)) = (0, 0)$ means: $x^2 = 1$ and $y^2 = 1$. Thus the gradient vanishes only at the points $(\pm 1, \pm 1)$. But these points do not satisfy $f(x, y) = 0$. Thus, the intersection is a smooth curve.
Solution 2. We apply the implicit function theorem to the system of equations
$\begin{matrix} x^3 + y^3 + 3z = 0 \\ x + y + z = 10 \end{matrix}$. To ensure a C^1 solution it is enough to check that the matrix of the first derivatives is non-degenerate at each point of the solution. Indeed,
suppose $rank \begin{pmatrix} 3x^2 & 3y^2 & 3 \\ 1 & 1 & 1 \end{pmatrix} < 2$. Then the first row of the Jacobi matrix is linearly dependent on the second row, i.e., $x^2 = y^2 = 1$. But the points $(\pm 1, \pm 1, z)$ do not solve the system. Thus the intersection is a smooth curve.
Remark Studying the intersection of the two surfaces by considering the 0-set of certain (linear – or other) combinations of the two equations will not, in general work. First, the 0-set of such combinations will, as a rule, be larger than the intersection of the two surfaces (in general, it will be a surface itself). Smoothness of that 0-set will not, therefore, imply (at least not without some argument) smoothness of the desired curve. Even if we can produce a combination whose 0-set is precisely the curve we are interested in, e.g., $(x+y+z-10)^2+(x^3+y^3+3z)^2$ it may have degenerate points that are harder to study than in the original problem. In the above example the 0-set of the function (which is the set we are interested in) is contained in the 0-set of the differential, so we cannot (directly) use the implicit function theorem.

(b) Solution 1.
Suppose $\nabla(f(0, 0)) \neq (0, 0)$ then, by implicit function theorem, the equation $f(x, y) = 0$ admits a smooth solution at $(0, 0)$, i.e. the curve $\{f(x, y) = 0\}$ is smooth
at $(0, 0)$. But the prescribed curve $\{(x, |x|),\ x \in [-1, 1]\}$ is not smooth.
Solution 2. Restrict the function $f(x, y)$ to the level curve, we get the function of one variable, $f(x, \phi(x))$. This function is constant, as we are on
the level curve. Thus the total derivative of this function is zero. Note that $\phi(x)$ is C^1 for $x \neq 0$. Therefore we have:

- $\partial_x f|_{(x,\phi(x))} + \partial_y f|_{(x,\phi(x))} = 0$ for $x > 0$
- $\partial_x f|_{(x,\phi(x))} - \partial_y f|_{(x,\phi(x))} = 0$ for $x < 0$.

Recall that $f(x, y)$ is C^1, thus in these two equations we can take the limits $x \to 0^+$ and $x \to 0^-$.
Then we get in the limit: $\partial_x f|_{(0,0)} = 0 = \partial_x f|_{(0,0)}$.
Solution 3. As f is C^1 write Lagrange expansion at $(0, 0)$: $f(x, y) = f(0, 0) + grad(f)|_c \cdot (x, y)$.

- For any $\epsilon > x > 0$ we have $f(x, x) = f(0, 0)$. Thus $grad(f)|_c \cdot (x, x) = 0$ and by taking limit $x \to 0^+$ we get: $grad(f)|_{(0,0)} \cdot (1, 1) = 0$.
- For any $-\epsilon < x < 0$ we have: $f(-x, x) = f(0, 0)$. Thus $grad(f)|_c \cdot (-x, x) = 0$ and by taking limit $x \to 0^-$ we get: $grad(f)|_{(0,0)} \cdot (-1, 1) = 0$.

Together this gives $grad(f)|_{(0,0)} = (0, 0)$.