Introduction to Algebraic Curves

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Homework 0

Spring 2018 (D.Kerner)

(1) Consider the subsets $Span_{\mathbb{R}}\Big((1,i),(i,1)\Big) \subset \mathbb{C}^2$, $Span_{\mathbb{C}}\Big((1,i),(i,1)\Big) \subset \mathbb{C}^2$. Which of them is a vector space over $\mathbb{R}$? Which of them is a vector space over $\mathbb{C}$? Which of them is invariant under the conjugation map, $(z_1,z_2) \to (\bar{z}_1,\bar{z}_2)$?

(2) (a) Which of the following sets are open/closed/path-connected/compact? In each case identify the interior, the closure and the boundary.
i. $\{|x|+|y| \le 1\} \subset \mathbb{R}^2$, ii. $\{|x|-|y| > 1\} \subset \mathbb{R}^2$, iii. $\Big([-1,1]\cap\mathbb{Q}\Big) \times \Big([-1,1]\cap\mathbb{Q}\Big) \subset \mathbb{R}^2$,
iv. $\{|z|<1\} \subset \mathbb{C}$, v. $\{z^6=1\} \subset \mathbb{C}$, vi. $\{z^2+w^2=1\} \subset \mathbb{C}^2$.
(b) Prove: any union of open subsets of $\mathbb{R}^n$ is open; any intersection of closed subsets of $\mathbb{R}^n$ is closed.
(c) Prove: any finite union of closed subsets of $\mathbb{R}^n$ is closed; any finite intersection of open subset is open. Why do we need the finiteness assumption here?
(d) Fix $p(z_1,z_2,\dots,z_n) \in \mathbb{C}[z_1,\dots,z_n]$. Prove that the functions $Re(p(\underline{z}))$, $Im(p(\underline{z}))$, $|p(\underline{z})|$ are continuous.

(3) (Implicit function theorem) At which points is the following curve (non)smooth? $\{y^2=(x^2-1)^3x\} \subset \mathbb{R}^2$?

A real/complex plane algebraic curve is the subset $C=\{f(x,y)=0\}$ of $\mathbb{R}^2$ (or $\mathbb{C}^2$), where $f(x,y) \in \mathbb{R}[x,y]$ (or $f(x,y) \in \mathbb{C}[x,y]$). The degree of the curve is defined as the degree of the polynomial $f(x,y)$, which is the largest among the total degrees of monomials participating in $f(x,y)$. For example, $deg(x^4+x^2y^3+y-1)=5$. Below $\Bbbk$ denotes $\mathbb{R}$ or $\mathbb{C}$.

(1) (a) Define the group action of $\Bbbk^\times := \Bbbk \setminus \{0\}$ on the set of (real or complex) plane curves by $\{f(x,y)=0\} \to \{\lambda \cdot f(x,y)=0\}$. Check that the action is trivial, each curve is preserved pointwise.
(b) The group $GL(2,\Bbbk)$ acts on the plane ($\Bbbk^2$) by linear transformations. The group $\Bbbk^2$ acts on the plane by parallel translations. Prove that together these groups generate the semi-direct product, $(GL(2,\Bbbk),\Bbbk^2) = \Bbbk^2 \rtimes GL(2,\Bbbk)$. (recall: $G=G_1 \rtimes G_2$ if G is generated by G_1,G_2, and $G_1 \cap G_2 = \{1\}$ and $G_1 \lhd G$.)
(c) Write down the action of the group $\Bbbk^\times \times \Bbbk^2 \rtimes GL(2,\Bbbk)$ on the set of algebraic curves. (i.e. what this action does to the equation $\{f(x,y)=0\}$.) Check that this action preserves the degree of curves.

(2) (Lines in the plane, i.e. curves of degree=1)
(a) Prove that any line in $\Bbbk^2$ can be brought, by the action of $\Bbbk^2 \rtimes GL(2,\Bbbk)$, to the line $\{y=0\}$.
(b) Associate to the line $\{a_xx+a_yy=b\} \subset \Bbbk^2$ the ring $\Bbbk[x,y]/(a_xx+a_yy-b)$. Prove: $\Bbbk[x,y]/(a_xx+a_yy-b) \approx \Bbbk[t]$. What is the geometric meaning of this ismorphism?
(c) Try to visualize/imagine the line $\{a_xx+a_yy=1\} \subset \mathbb{C}^2$.

(3) (Plane conics, i.e. curves of degree=2)
(a) Let $f(x,y) \in \mathbb{R}[x,y]$ be of degree 2. Prove that by linear transformations (the group $GL(2,\mathbb{R})$), the parallel translations (the group $\mathbb{R}^2$), and the scaling ($f(x,y) \to \lambda f(x,y)$, for some $\lambda \in \mathbb{R}\setminus\{0\}$) the curve $\{f(x,y)=0\} \subset \mathbb{R}^2$ can be brought to one (and only one) of the following forms: $y^2 \pm x^2 = 0$, $y^2 \pm x^2 = 1$, $x^2+y^2=-1$, $y=x^2$, $x^2=1$, $x^2=0$. Draw the corresponding curves. (These are called: canonical forms of plane conics.) Which of the curves are smooth?
(b) Let $f(x,y) \in \mathbb{C}[x,y]$ be of degree 2. What are the canonical forms in this case?
(c) Consider the rings $\Bbbk[x,y]/x^2-y^2$, $\Bbbk[x,y]/x^2-y^2+1$, $\Bbbk[x,y]/x^2-1$, $\Bbbk[x,y]/x^2$. Which of them are integral domains? Rings without nilpotents? PID?

(4) (Plane cubics) Trace the change of the curve $\{y^2=x^3+x^2+\epsilon\} \subset \mathbb{R}^2$ for $\epsilon \in (-1,1)$. What happens at $\epsilon=0$? (You can use the computer. Google: "plane cubic", "nodal cubic")
Trace the change of the curve $\{y^2=x^3+\epsilon \cdot x^2\} \subset \mathbb{R}^2$ for $\epsilon \in (-1,1)$.

(5) (a) Let $C=\{x^2+y^2=1\} \subset \Bbbk^2$, where $\Bbbk \in \mathbb{R},\mathbb{C}$. Let $C \stackrel{\pi_x}{\to} \Bbbk^1$ be the projection onto the $\hat{x}$-axis. Describe the image, $\pi_x(C)$. How many preimages has a point $x \in \pi_x(C)$?
(b) Let $C=\{f(x,y)=0\} \subset \mathbb{C}^2$ for some polynomial $f(x,y) \in \mathbb{C}[x,y]$, of degree d. Prove that the image of the projection π_x is either a finite set of points or the $\hat{x}$-axis with a finite set of points removed. (e.g. $y(x^2-1)=1$.) Suppose for every point x of the $\hat{x}$-axis the number of preimages (in C) is finite. Prove that this number is generically d. (i.e. for all values of x, except for a finite subset, this number equal d.)