Introduction to Algebraic Curves

201.2.4451. Spring 2018 (D.Kerner)

Homework 1

(1) (a) Which of the following functions are holomorphic? (Where?)
  i. $f(x,y) = x^2 - y^2 - 2ixy$, ii. $f(z) = \frac{1}{Im(z)+i}$, iii. $f(x,y) = log(x^2+y^2) + 2i \cdot arctan\frac{y}{x}$.
(b) Suppose $f(z), h(z)$ are holomorphic and $f(z) \cdot g(x,y) = h(z)$ holds on $\mathbb{C}$. Prove that $g(x,y)$ is holomorphic at all points where $f(z) \neq 0$.
(c) Obtain the polar form of Cauchy-Riemann equations, for $z = r \cdot e^{i\phi}$: $\frac{\partial f}{\partial \phi} = i \cdot r \cdot \frac{\partial f}{\partial r}$.

(2) (a) Fix a real manifold X and two open sets with homeomorphisms, $X \supset \mathcal{U}_i \stackrel{\phi_i}{\tilde{\to}} \phi_i(\mathcal{U}_i) \subset \mathbb{R}^n$, $i = 1, 2$. Suppose $\phi_1 \circ \phi_2^{-1}$ is smooth (C^∞). Does this imply that $\phi_2 \circ \phi_1^{-1}$ is smooth?
(b) Fix a Riemann surface X and two open sets with homeomorphisms, $X \supset \mathcal{U}_i \stackrel{\phi_i}{\tilde{\to}} \phi_i(\mathcal{U}_i) \subset \mathbb{C}$, $i = 1, 2$. Suppose $\phi_1 \circ \phi_2^{-1}$ is holomorphic. Does this imply that $\phi_2 \circ \phi_1^{-1}$ is holomorphic? (You can use the 'normal form of holomorphic function' theorem: if $f(z)$ is holomorphic and $f'(0) = 0 = \cdots = f^{(n-1)}(0) \neq f^{(n)}(0)$, then, after a local change of coordinates holds: $f(z) = z^n$.)
(c) Fix a curve $C = \{f(z,w) = 0\} \subset \mathbb{C}^2$. Denote by π_z, π_w the projections onto the coordinates axes, $\mathbb{C}_z, \mathbb{C}_w \subset \mathbb{C}^2$. Suppose for an open subset $\mathcal{U} \subset C$ the restrictions $\pi_z|_{\mathcal{U}}$, $\pi_w|_{\mathcal{U}}$ are homeomorphisms. Are the maps $\pi_z \circ \pi_w^{-1}$, $\pi_w \circ \pi_z^{-1}$ necessarily holomorphic?

(3) Let X be a real/complex manifold with some (compatible) charts $\{(\mathcal{U}_i, \phi_i)\}$.
(a) Fix a collection of invertible maps, $\Bbbk^n \stackrel{\psi_i}{\to} \Bbbk^n$ which are C^∞-diffeomorphisms/bi-holomorphic. Prove that the collection $\{(\mathcal{U}_i, \phi_i), (\mathcal{U}_i, \psi \circ \phi_i)\}$ is also a set of compatible charts.
(b) Fix any open subsets $\mathcal{V}_i \subset \mathcal{U}_i$. Prove that the collection of pairs $\{(\mathcal{U}_i, \phi_i), (\mathcal{V}_i, \phi_i|_{\mathcal{V}_i})\}$ is also a set of compatible charts.

(4) Cover $S^{2n} = \{x_1^2 + \cdots + x_{2n+1}^2 = 1\} \subset \mathbb{R}^{2n+1}$ by two open subsets: $\mathcal{U}_{down} = \{x_{2n+1} \neq 1\}$, $\mathcal{U}_{up} = \{x_{2n+1} \neq -1\}$.
(a) Introduce the complex coordinates on $\mathcal{U}_*$ by projecting from the poles onto the hyperplane $x_{2n+1} = 0$, and identifying: $\{z_j = x_{2j-1} + ix_{2j}\}$. Do these charts define a real/complex manifold structure on S^{2n}?
(b) Introduce the complex coordinates on $\mathcal{U}_*$ by projecting from the poles onto the hyperplanes $x_{2n+1} = \pm 1$, and identifying: $\{z_j = x_{2j-1} + ix_{2j}\}$ for $x_{2n+1} = 1$ and $\{w_j = x_{2j-1} - ix_{2j}\}$ for $x_{2n+1} = -1$. For which n do these charts define a real/complex manifold structure on S^{2n}?

(5) We have proved in the class:
(a) Riemann surfaces are orientable (when considered as real manifolds);
(b) the complex version of Implicit Function Theorem;
(c) $S^2 \subset \mathbb{R}^3$ has a Riemann surface structure;
(d) $\mathbb{CP}^1$ is a compact Riemann surface.
Go over all the details of the proofs. Prove that S^2 with the Riemann surface structure is biholomorphic to $\mathbb{CP}^1$. (Construct an explicit biholomorphism)

(6) Prove that plane affine complex-algebraic curves, $\{f(z,w) = 0\} \subset \mathbb{C}^2$ (with f a polynomial), are never compact. (You can use question 5 of hwk.0)

(7) We have constructed complex tori in the class, $\mathbb{C}/_L$.
(a) Prove that any complex torus is topologically a (real, dimension two) compact manifold of genus 1. (Construct an explicit homeomorphism of a complex torus to $S^1 \times S^1$.)
(b) Prove that any complex torus is an abelian group. Prove that this is an abelian group with division, i.e. for any $x \in \mathbb{C}/_L$ and $n \in \mathbb{N}$ exists $y \in \mathbb{C}/_L$ satisfying: $n \cdot y = x$. How many such y there exist for a fixed pair (x, n)?

(8) Let $\Bbbk$ be one of $\mathbb{R}$, $\mathbb{C}$ and define $X = \{y^2 = x^3\} \subset \Bbbk^2$. Denote by π the projection from $(0,0)$ to the line $\{x = 1\}$.
(a) Prove that we get a homeomorphism $X \setminus (0,0) \stackrel{\pi}{\tilde{\to}} \Bbbk^1 \setminus \{0\}$. (Write the explicit formulae.) Check that π defines the structure of (real/complex) manifold on the topological space $X \setminus (0,0)$.
(b) Could one use, instead of $(0,0)$, some other point of $\Bbbk^2$?
(c) Prove that the in (a) homeomorphism extends uniquely to the homeomorphism $X \stackrel{\pi}{\tilde{\to}} \Bbbk^1$. (One might be tempted to use π to define a manifold structure on X, but this is not the structure that comes from the embedding $X \subset \Bbbk^2$.)