Introduction to Algebraic Curves

201.2.4451. Spring 2018 (D.Kerner)

Homework 2

(1) Consider two collections of complex charts on $\mathbb{R}^2$:
   (a) For any open $\mathcal{U} \subset \mathbb{R}^2$ take $\phi_U(x,y) = x + iy \in \mathbb{C}$.
   (b) For any open $\mathcal{U} \subset \mathbb{R}^2$ take $\phi_U(x,y) = (x+iy)g(x,y) \in \mathbb{C}$. (Here $g(x,y)$ is a complex valued function, the same for all the open sets.)

For which functions $g(x,y)$ are these collections of charts compatible?

(2) Let $\Bbbk$ be one of $\mathbb{R}$, $\mathbb{C}$.
   (a) In the class we have defined the (real/complex) projective plane and proved that this is a compact connected (real/complex) manifold. Go over all the details of the proof.
   (b) Prove: any two distinct lines in $\mathbb{P}^2_{\Bbbk}$ intersect in precisely one point. (What happens to the parallel lines of $\Bbbk^2$?)
   (c) Let $f(z,w)$ be a polynomial of degree d. Define $F(x_0,x_1,x_2) := x_0^d f(\frac{x_1}{x_0}, \frac{x_2}{x_0})$. Prove that d here is the minimal degree to ensure that F is a polynomial. Prove that d here is the maximal degree to ensure that F is not divisible by x_0.
   (d) In the class, when proving that a complex smooth projective plane curve is a Riemann surface, we did not check the holomorphicity of transition maps. Check this.
   (e) An algebraic curve (in $\Bbbk^2$ or $\mathbb{P}^2$) is called reducible if its defining equation factorizes non-trivially, i.e. $F = F_1 \cdot F_2$, where $F_i \neq const$ and $gcd(F_1,F_2) = 1$. (Otherwise the curve is called irreducible.) How the (ir)reducibility of $X \subset \Bbbk^2$ is related to (ir)reducibility of $\overline{X} \subset \mathbb{P}^2$?
   (f) Fix a projective algebraic plane curve $X \subset \mathbb{P}^2_{\mathbb{C}}$. Prove that every projective line $L \subset \mathbb{P}^2_{\mathbb{C}}$, has a non-empty intersection with X. Can we replace $\mathbb{C}$ by $\mathbb{R}$ here? Can we replace $\mathbb{P}^2_{\mathbb{C}}$ by $\mathbb{C}^2$ here?

(3) Let $\{X_i = \{f_i(x,y) = 0\} \subset \Bbbk^2\}_{i=1,2}$ be two plane curves (C^∞ or holomorphic), smooth at some point $p \in \Bbbk^2$. Define their degree of tangency at p as the vanishing order: let z_1 be the local coordinate on X_1, then $i_p(X_1,X_2) := ord(f_2(x(z_1), y(z_1)))$. (Another name for this number is: the intersection multiplciity. Note, this can be finite or infinite.)
   (a) Compute the degree of tangency at $p = (0,0)$ for the curves $X_1 = \{y + x = x^n\}$, $X_2 = \{y + x = -x^n\}$.
   (b) Prove that $i_p(X_1,X_2)$ does not depend on the choice of the local coordinate of X_1 at p.
   (c) Note that the locally defining equations of X_1, X_2 are non-unique. If u_1, u_2 are (C^∞/holomorphic) functions that do not vanish at p then locally $X_i = \{u_i(x,y) \cdot f_i(x,y) = 0\}$. Prove that $i_p(X_1,X_2)$ does not depend on the choice of the locally defining equation.
   (d) Prove that $i_p(X_1,X_2)$ does not depend on the choice of local coordinates in $\Bbbk^2$.
   (e) One would like to know that this number is independent of the order of the curves, i.e. $i_p(X_1,X_2) = i_p(X_2,X_1)$. We will prve this later.

(4) (a) (Tautology) Let X be a Riemann surface with a chart $X \supseteq \mathcal{U} \xrightarrow{\phi} \phi(\mathcal{U}) \subseteq \mathbb{C}$. Prove that ϕ is a holomorphic function on $\mathcal{U}$.
   (b) Prove that $\mathcal{O}_X(\mathcal{U})$ is a commutative ring that contains $\mathbb{C}$. Fix a point $p \in \mathcal{U}$ and consider the ideal $I_p = \{f \in \mathcal{O}_X(\mathcal{U}),\ f(p) = 0\}$. Prove that this ideal is maximal and principal, find a generator.
   (c) Let f be a complex valued (though not necessarily holomorphic) function on $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
   Prove: f is holomorphic at $z = \infty$ iff $f(\frac{1}{z})$ holomorphic at $z = 0$.
   (d) (i) Prove: the only holomorphic functions on $\mathbb{CP}^1$ are constants.
   (ii) Prove: the only holomorphic functions on a complex torus are constants.
   (e) Suppose a Riemann surface X is defined as a subset of $\mathbb{C}^n$ by a system of holomorphic equations. Prove that any holomorphic function $\mathbb{C}^n \xrightarrow{f} \mathbb{C}$ restricts to a holomorphic function $X \xrightarrow{f|_X} \mathbb{C}$. Prove that X is not isomorphic to $\mathbb{CP}^1$, neither to a complex torus.
   (f) Fix two homogeneous polynomials of the same degree, $p(z_0,z_1), q(z_0,z_1) \in \mathbb{C}[z_0,z_1]$. Prove that $\frac{p(z_0,z_1)}{q(z_0,z_1)}$ defines a holomorphic function at all the points of $\mathbb{P}^1_{\mathbb{C}}$ where q does not vanish.
   (g) Fix a Riemann surface $X = \{F(z_0,z_1,z_2) = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ and two homogeneous polynomials of the same degree, $p(z_0,z_1,z_2), q(z_0,z_1,z_2) \in \mathbb{C}[z_0,z_1,z_2]$. Prove that $\frac{p(z_0,z_1,z_2)}{q(z_0,z_1,z_2)}$ defines a holomorphic function at all the points of X where q does not vanish.