Introduction to Algebraic Curves

201.2.4451. Spring 2018 (D.Kerner)

Homework 3

(1) (a) In the class we have defined the notion of a (C^∞/holomorphic) map of (real/complex) manifolds. This definition used choices of particular charts. Prove the independence of these notions of all the choices.
(b) Let $X = \{y^2 = x^7 + 1\} \subset \mathbb{C}^2$. Which of the following functions restrict to a holomorphic/meromorphic function on X? i. $f(x,y) = tan(\frac{y}{x})$, ii. $f(x,y) = e^{\frac{y-1}{x}}$.
(c) Let f be a meromorphic function on a Riemann surface X. Prove that f defines a holomorphic map $X \xrightarrow{f} \mathbb{P}^1_{\mathbb{C}}$.
(d) Fix a lattice $L \subset \mathbb{C}$ and the corresponding complex torus, $\mathbb{C}/_L \xrightarrow{\pi} X$. Prove that π is a holomorphic map of Riemann surfaces. Prove: a function $X \xrightarrow{f} \mathbb{C}$ is holomorphic/meromorphic iff the function $f \circ \pi$ is of this type.

(2) Prove that the composition of holomorphic maps of Riemann surfaces is holomorphic. (This defines the map $Maps^{hol}(X,Y) \times Maps^{hol}(Y,Z) \to Maps^{hol}(X,Z)$.) In particular:
(a) The composition of a holomorphic map and a holomorphic function is holomorphic.
(b) The composition of a holomorphic map and a meromorphic function is meromorphic.
(c) The map $X \xrightarrow{\phi} Y$ induces the homomorphisms of rings $\mathcal{O}_Y(\mathcal{U}) \xrightarrow{\phi^*} \mathcal{O}_X(\phi^{-1}(\mathcal{U}))$, and $\mathcal{M}_Y(\mathcal{U}) \xrightarrow{\phi^*} \mathcal{M}_X(\phi^{-1}(\mathcal{U}))$, by $f \to \phi^*(f) = f \circ \phi$.

(3) (a) Suppose two holomorphic maps of compact Riemann surfaces, $X \overset{f,g}{\to} Y$, coincide on infinite set of points. Prove that they coincide on X.
(b) Let $X \xrightarrow{f} Y$ be a non-constant holomorphic map. Prove that for any $y \in Y$ the set of preimages, $f^{-1}(y)$ is discrete.
(c) Go over all the details of the proof of local normal form for a holomorphic map.
(d) Suppose a holomorphic map $X \xrightarrow{f} Y$ is a bijection of sets. Prove that it is holomorphically invertible.
(e) Prove that the multiplicity of a map at a point does not depend on the choice of local coordinates.
(f) Let $X \xrightarrow{f} Y$ a holomorphic map, with X compact. Prove: $mult_x(f) > 1$ only for a finite number of points.
(g) Let f be a non-constant meromorphic function on a compact Riemann surface. Prove: $\sum\limits_{x \in X} ord_x(f) = 0$.

(4) (a) Prove that any holomorphic map $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ is presentable as $(z_0 : z_1) \to (p(z_0, z_1) : q(z_0, z_1))$, for some homogeneous polynomials of the same degree d. What is the degree of this map?
(b) An invertible holomorphic map $X \xrightarrow{f} X$ is called an automorphism. (Prove that f is an automorphism iff its degree is 1.) The set of all the automorphism of a Riemann surface is a group, $Aut(X)$.
(c) Fix a matrix $A \in GL(2, \mathbb{C})$. Prove that the map $(z_0 : z_1) \to (z_0 : z_1)A$ defines an automorphism of $\mathbb{P}^1$. Prove that this correspondence defines a surjective homomorphism of groups $GL(2, \mathbb{C}) \twoheadrightarrow Aut(\mathbb{P}^1)$. What is the kernel?
(d) Fix a compact Riemann surface X and a meromorphic function f. Prove that f has at least one zero and at least one pole. Prove: if f has precisely one pole (or zero) then it is an isomorphism $X \overset{f}{\xrightarrow{\sim}} \mathbb{P}^1$.

(5) (a) Let $X \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth projective curve of degree d. Fix a point $p \notin X$ and a projective line $L \subset \mathbb{P}^2$ that does not pass through p. Define the map $X \xrightarrow{\pi} L \approx \mathbb{P}^1$ by projecting from p to the line. Prove that π is a holomorphic map of Riemann surfaces. Compute the degree of this map.
(b) Prove that every projective smooth curve of degree 2 is isomorphic to $\mathbb{P}^1$. (Hint: fix any point $p \in X$ and a line $p \notin L \subset \mathbb{P}^2$ and project from p to L. Compute the degree of this projection.)
(c) Consider the projection of $X = \{y = x^d\} \subset \mathbb{C}^2$ onto the axis $\mathbb{C}_x$. Is this an isomorphism of Riemann surfaces? Does this extend to an isomorphism $\mathbb{P}^2 \supseteq \overline{X} \to \mathbb{P}^1$?
(d) Fix a smooth projective curve $X = \{f(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2$, with $f(1,0,0) \neq 0$. Prove that the projection π_{x_1,x_2} (from $(1:0:0)$ to the line $(x_0 = 0)$) defines the holomorphic map $X \to \mathbb{P}^1$. Prove that $p \in X$ is a ramification point of this map iff $\partial_0 f|_p = 0$.
(e) Write down the full ramification data (the ramification points and the relevant multiplicities) for the projection $\pi_{x_1 x_2}$ of the curve $\{x_0^d + x_1^d + x_2^d = 0\} \subset \mathbb{P}^2$.
(f) Consider the map $\mathbb{C} \to \mathbb{C}^d$, $x \to (x, x^2, \ldots, x^d)$. Prove that its image, X, is a Riemann surface. Let $X \xrightarrow{\pi_j} \mathbb{C}$ be the projection onto j'th axis. Prove that π_j is a holomorphic map, compute its degree, and find the full ramification data.