Introduction to Algebraic Curves

201.2.4451. Spring 2018 (D.Kerner)

Homework 9

(1) (a) An algebraic subset $X \subset \mathbb{P}^n$ is called a (projective) plane if $I_X \subset \Bbbk[\underline{x}]$ is generated by linear forms.
For any element $\phi \in \mathbb{P}GL(n+1)$ prove: $\phi(X)$ is a projective plane (i.e. $\phi^*(I_X)$ is ...).
(b) Check: any projective plane is the projective closure of an affine plane.
(c) Fix some generating linear form, $I_X = (l_1(\underline{x}), \dots, l_r(\underline{x}))$, and take the matrix of their coefficients, $A \in Mat_{r\times(n+1)}(\Bbbk)$. The codimension of $X \subseteq \mathbb{P}^n$ is the row-rank of this matrix, while the dimension is $n - codim(X)$. Check that these do not depend on the choice of coordinates in $\mathbb{P}^n$, generators of I_X. Check that the (co)dimension of a linear subset coincides with the (co)dimension of (any of) its affine part.
(d) Fix an affine plane of dimension r with the parametrization $L = \{\vec{v}_0 + \sum t_i\vec{v}_i | \ (t_1, \dots, t_r) \in \Bbbk^r\} \subset \Bbbk^n$.
Convert this into the parametrization of $\bar{L} \subset \mathbb{P}^n$.
(e) Fix some planes $X, Y \subset \mathbb{P}^n$. Prove: $dim(X \cap Y) \geq dim(X) + dim(Y) - n$.
(f) Prove: any three points $pt_1, pt_2, pt_3 \in \mathbb{P}^2$ not lying on one line can be brought by $\mathbb{P}GL(3)$ to $[1:0:0]$, $[0:1:0]$, $[0:0:1]$. Formulate and prove the higher dimensional analogue of this statement.
(g) Prove: any three lines in $\mathbb{P}^2$ that do not all pass through one point can be brought to the lines $\{x_i = 0\}_{i=0,1,2}$. Formulate and prove the analogous statement for hyperplanes in $\mathbb{P}^n$.

(2) (a) Prove: any homogeneous ideal in $\Bbbk[\underline{x}]$ is homogeneously finitely generated.
(b) For a projective algebraic set $X \subset \mathbb{P}^n$ take the coordinate ring $\Bbbk[X]$. Prove: every element of $\Bbbk[X]$ can be written (uniquely) as $\sum \bar{f}_i$, where $\bar{f}_i \in \Bbbk[X]$ is the image of some homogeneous polynomial $f_i \in \Bbbk[x_0, \dots, x_n]$
Prove: the images of degree-d-homogeneous polynomials form a finite dimensional vector subspace $\Bbbk[X]_d \subset \Bbbk[X]$.
(c) Fix a hypersurface $X = \{f(\underline{x}) = 0\} \subset \mathbb{P}^n$. Check the exactness of the sequence $0 \to \Bbbk[\underline{x}] \xrightarrow{\times f} \Bbbk[\underline{x}] \to \Bbbk[X] \to 0$.
Compute $dim_{\Bbbk}\Bbbk[X]_d$.
(d) Take some algebraic subsets $X \subseteq Y \subseteq \mathbb{P}^n$, with X an irreducible hypersurface.
Prove: either $X = Y$ or $Y = \mathbb{P}^n$ or Y is reducible.

(3) (a) Prove: the projective closure of an affine hypersurface is a hypersurface, i.e. it is defined by one equation.
(b) Prove: $X \subset \mathbb{P}^n$ is a projective hypersurface iff for any affine chart $\mathcal{U}_i$: $X \cap \mathcal{U}_i$ is an affine hypersurface.
(c) Let $I = (x_1 + x_3^2, x_2 + x_3^2)$ and $X = V(I) \subset \Bbbk^3$. Describe $\overline{X} \subset \mathbb{P}^3$ and $I_{\overline{X}}$.
(Warning: the defining ideal of $\overline{X} \subset \mathbb{P}^3$ is not generated by just two elements.)
(d) In the lecture we have obtained the local ring $\mathcal{O}_{(X,pt)}$ of $X \subset \mathbb{P}^n$ by first restricting to some affine chart $\mathcal{U}_i$ and then localizating. Prove that $\mathcal{O}_{(X,pt)}$ does not depend on the choice of the affine chart.
(e) In the lecture we have defined the function field $\Bbbk(X)$. Check that this is indeed a field. For $X = \mathbb{P}^n$ compute its transcendence degree over $\Bbbk$.
(f) Let $X \subset \mathbb{P}^n$ be defined by a prime ideal $I \subset \Bbbk[\underline{x}]$ and let $f \in \Bbbk(X)$. Prove: the sets of zeros/poles of f are algebraic subsets of $\mathbb{P}^n$.

(4) (a) Suppose $\Bbbk = \bar{k}$. Show that any smooth conic is $\mathbb{P}GL(2)$-equivalent to $V(x^2 + y^2 + z^2)$.
(b) Show that pt is a singular point of $\{f(x,y,z) = 0\} \subset \mathbb{P}^2$ iff $\partial_x f|_{pt} = \partial_y f|_{pt} = \partial_z f|_{pt} = 0$. (Assume $char(\Bbbk) \nmid deg(f)$.)
If $[x_0 : y_0 : z_0]$ is a smooth point, check that the tangent line is defined by $(x - x_0)\partial_x f + (y - y_0)\partial_y f + (z - z_0)\partial_z f = 0$.
(c) For any $pt \in C = \{f(x,y,z) = 0\} \subset \mathbb{P}^2$ prove: $mult_{pt}(C) \leq mult_{pt}\{\partial_x f(x,y,z) = 0\} + 1$.
(d) Find all the intersection points and the local intersection multiplicities of $V(y^2z - x(x - 2z)(x + z))$, $V(y^2 + x^2 - 2xz)$ in $\mathbb{P}^2$.
(e) (Here $char(\Bbbk) = 0$.) Suppose the curve $C = V(f) \subset \mathbb{P}^2$ is irreducible. Prove: $\partial_x f|_C \cdot \partial_y f|_C \cdot \partial_z f|_C \not\equiv 0$.
In particular C has a finite number of singular points.
(f) (Here $\Bbbk = \bar{k}$) Prove: a plane cubic with more than one singular point is reducible.
Classify, up to $\mathbb{P}GL(3)$-equivalence, all the reducible plane cubics.

(5) Fix a local ring $(R, \mathfrak{m})$. The order of an element $f \in R$ is $ord(f) := sup\{j | \ f \in \mathfrak{m}^j\} \leq \infty$. (Dis)prove the following properties. Even if a property fails in general, give several examples of local rings where this property holds.
i. $ord(f) = \infty$ iff $f = 0$. ii. $ord(f \pm g) \geq min(ord(f), ord(g))$. iii. $ord(f \cdot g) = ord(f) + ord(g)$
iv. $ord(\frac{f}{g}) = ord(f) - ord(g)$.