Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 0

This is a warm-up on some basics that wil be assumed during the course.
All the questions should be solved *before* the beginning of the semester.

$\mathbb{R}^n$ is always considered with the standard topology. All the subsets of $\mathbb{R}^n$ are considered with the induced topology. For a subset $X \subset \mathbb{R}^n$ denote the interior by $Int(X)$. Denote the homeomorphism of topological spaces by $X \overset{homeo}{\approx} Y$. Denote the unit matrix by $\mathbb{1}$, the zero matrix by $\mathbb{O}$. $Mat_{m\times n}(\Bbbk)$ denotes the space of $m \times n$ matrices over a field $\Bbbk$. Usually $\Bbbk = \mathbb{R}, \mathbb{C}$, then, topologically, $Mat_{m\times n}(\Bbbk) \overset{homeo}{\approx} \Bbbk^{mn}$. $Ball$ = an open ball in $\mathbb{R}^n$, D^n = a closed disc.

(1) (a) Let $X \subset \mathbb{R}$ be one of the sets $\mathbb{Z}, \mathbb{Q}$, $[0,1]$, $[0,1] \setminus \Big(\underset{k\geq 1}{\cup} (\frac{1}{3^k}, \frac{2}{3^k}) \Big)$. Describe the sets $\overline{X}$, $Int(X)$, ∂X.

(b) Fix some subsets of a topological space, $X, Y \subset Z$. Prove/disprove by a counterexample: $\overline{X \cap Y} = \bar{X} \cap \bar{Y}$, $\overline{X \cup Y} = \bar{X} \cup \bar{Y}$, $\overline{X \times Y} = \bar{X} \times \bar{Y}$, $Int(X \cap Y) = Int(X) \cap Int(Y)$, $Int(X \cup Y) = Int(X) \cup Int(Y)$, $Int(Int(X)) = Int(X)$, $\overline{\overline{X}} = \bar{X}$, $\overline{Int(X)} = Int(\overline{X})$, $\partial(X \times Y) = \Big(\partial(X) \times Y\Big) \coprod \Big(X \times \partial(Y)\Big)$.

(c) Prove/disprove by a counterexample

(i) If $X \subset \mathbb{R}^n$ is a discrete topological subspace then $|X| \leq \aleph_0$. (What about the converse?)

(ii) Let $X, Y \subset \mathbb{R}^n$ be finite or countable subsets. Then $X \overset{homeo}{\approx} Y$ iff $|X| = |Y|$.

(iii) If X is connected and the map $X \overset{\phi}{\to} Y$ is continuous then $\phi(X)$ is connected.

(iv) Compactness is preserved under homeomorphisms.

(v) The number of connected components is preserved under homeomorphisms.

(vi) The map $X \overset{(f_1, f_2)}{\to} Y_1 \times Y_2$ is continuous iff f_1, f_2 are continuous.

(vii) If $\mathcal{U} \overset{f}{\to} \mathbb{R}^n$ is a continuous function then its graph, $\Gamma_f \subset \mathcal{U} \times \mathbb{R}$, is homeomorphic to $\mathcal{U}$.

(2) (a) Classify the letters of English alphabet up to homeomorphism. (The topology is induced from $\mathbb{R}^2$.)

(b) Prove that the following sets are all homeomorphic:

$\mathbb{R}^2 \setminus \{0, 0\}$, $\quad \{0 < x^2 + y^2 < 1\}$, $\quad \left\{ \begin{array}{c} |x| < 1, |y| < 1 \\ (x,y) \neq (0,0) \end{array} \right\}$, $\quad \mathbb{R}^2 \setminus \{x^2 + y^2 \leq 1\}$.

(c) Prove that $\mathbb{R}^3 \supset \{1 < x^2 + y^2 + z^2 < 4\} \setminus \left\{ \begin{array}{c} 1 < x^2 + y^2 + z^2 < 4 \\ x^2 + y^2 \leq \frac{1}{10},\ z > 0 \end{array} \right\}$ is homeomorphic to the open 3-dimensional ball.

(d) Fix some numbers $k, a_1, \dots, a_n \in \mathbb{R}_{>0}$. Prove that the "distorted ball" $\{\sum\limits_{i=1}^{n} \frac{|x|^k}{a_i} \leq 1\} \subset \mathbb{R}^n$ is homeomorphic to the standard ball, $\{\sum\limits_{i=1}^{n} x_i^2 \leq 1\} \subset \mathbb{R}^n$. Prove that the standard ball is homeomorphic to the standard cube $[0,1]^n \subset \mathbb{R}^n$.

(e) Prove that the "distorted sphere" $\{\sum\limits_{i=1}^{n} \frac{|x|^k}{a_i} = 1\} \subset \mathbb{R}^n$ is homeomorphic to the standard sphere $S^{n-1} = \{\sum\limits_{i=1}^{n} x_i^2 = 1\} \subset \mathbb{R}^n$. Prove that S^{n-1} is homeomorphic to the boundary $\partial\Big([0,1]^n\Big)$.

(f) Prove: $S^n \setminus \{x\} \overset{homeo}{\approx} S^n \setminus D^n \overset{homeo}{\approx} Ball$ and $S^n \setminus \{x_1, x_2\} \overset{homeo}{\approx} S^{n-1} \times (0,1)$.

(3) (a) Prove that the following maps are continuous:

i. $Mat_{m\times m}(\mathbb{R}) \overset{(det, trace)}{\to} \mathbb{R}^2$, ii. $Mat_{m\times m}(\mathbb{R}) \overset{\phi}{\to} Mat_{m\times m}(\mathbb{R})$, $\phi(A) = A^T$,

iii. $Mat_{m\times m}(\mathbb{R}) \times Mat_{m\times m}(\mathbb{R}) \overset{\phi}{\to} Mat_{m\times m}(\mathbb{R})$, $\phi(A, B) = A + B$, $\phi(A, B) = AB$.

iv. $GL(m, \mathbb{R}) \overset{\phi}{\longrightarrow} GL(m, \mathbb{R})$, $\phi(A) = A^{-1}$.

(b) Let ,$\Bbbk = \mathbb{R}, \mathbb{C}$. The group $GL(m, \Bbbk) \subset Mat_{m\times m}(\Bbbk)$ gets the induced topology. Which of the following subgroups of $GL(m, \mathbb{R})$, $GL(m, \mathbb{C})$ are open/closed:

i. $SL(m) := \{A |\ det(A) = 1\}$, ii. $O(m) := \{A |\ AA^t = \mathbb{1}\}$, iii. $SO(m) := \{A |\ AA^T = \mathbb{1},\ det(A) = 1\}$, iv. $U(m) := \{A |\ A\bar{A}^t = \mathbb{1}\}$, v. $SU(m) := \{A |\ A\bar{A}^t = \mathbb{1},\ det(A) = 1\}$.

(c) Prove that $GL(m, \mathbb{R})$, $O(m, \mathbb{R})$ have precisely two path-connected components.

(d) Prove that $GL(m, \mathbb{C})$, $U(m, \mathbb{C})$ are path-connected.

(e) Denote by $\Sigma_{diag} \subset Mat_{m\times m}(\mathbb{C})$ the subset of diagonalizable matrices. Prove: $\overline{\Sigma_{diag}} = Mat_{m\times m}(\mathbb{C})$. (By adding small noise any matrix can be turned into diagonalizable.) Is Σ_{diag} an open subset?

(4) (a) Fix some finite subsets, $X, Y \subset \mathbb{R}^n$ with $|X| = |Y|$. Prove: $\mathbb{R}^n \setminus X \overset{homeo}{\approx} \mathbb{R}^n \setminus Y$.
(b) Establish the following homeomorphisms:
i. $\mathbb{R}^n \setminus \mathbb{R}^k \approx S^{n-k-1} \times \mathbb{R}^{k+1}$, here $\mathbb{R}^k = \{x_{k+1} = \cdots = x_n = 0\} \subset \mathbb{R}^n$.
ii. $S^n \cap \{\sum\limits_{i=1}^{k} x_i^2 \leq \sum\limits_{k+1}^{n} x_i^2\} \approx S^{k-1} \times D^{n-k+1}$; iii. $O(n) \overset{homeo}{\approx} SO(n) \times O(1)$;
iv. $GL(n) \approx SL(n) \times GL(1)$; v. $GL_+(n) \approx SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$

(5) (a) An embedded torus in $\mathbb{R}^3$ is a surface parameterized by

$$S^1 \times S^1 \ni (\phi_1, \phi_2) \to \Big((R + rsin(\phi_2))cos(\phi_1), (R + rsin(\phi_2))sin(\phi_1), Rcos(\phi_2)\Big) \in \mathbb{R}^3.$$

Draw this torus, identify R, r, the meridians ($\phi_1 = const$) and the parallels ($\phi_2 = const$). Consider the intersections of the torus with all the possible planes in $\mathbb{R}^3$. What are the possible topological types (up to homeomorphism) of the intersections?
(b) Prove that the product $\underbrace{S^1 \times \cdots \times S^1}_{n\ times}$ is embeddable into $\mathbb{R}^{n+1}$. (Write an explicit parametrization.)
(c) Fix $R > 1$ and $n \in \mathbb{Z}_{>0}$ and define the surface $S_n \subset \mathbb{R}^3$ by parametrization,

$$S^1 \times [-1, 1] \ni (\phi, t) \to \Big((R + t \cdot cos(\frac{n\phi}{2}))cos(\phi), (R + t \cdot cos(\frac{n\phi}{2}))sin(\phi), t \cdot sin(\frac{n\phi}{2})\Big)$$

(i) Draw/imagine these surfaces. (For $n = 0$, $n = 1$ these are very classical?)
(ii) Prove that all $\{S_n\}_{n \in 2\mathbb{N}}$ are homeomorphic and all $\{S_n\}_{n \in 2\mathbb{N}+1}$ are homeomorphic. (Here the homeomorphisms are of surfaces only, not of their embeddings.)

(6) Define the convex hull of a subset $X \subset \mathbb{R}^n$ as $Conv(X) := \big\{ \sum t_i \underline{a}_i | \ \{t_i \geq 0\},\ \sum t_i = 1,\ \{\underline{a}_i \in X\} \big\}$
(a) Check that any convex polygon in $\mathbb{R}^2$ is the convex hull. (What is the minimal system of generators?)
(b) Suppose X is open/closed/bounded/compact. Is $Conv(X)$ necessarily open/closed/bounded/compact?
(c) Let $L_X \subseteq \mathbb{R}^n$ be the plane of minimal dimension that contains X. Check: $L_X = L_{Conv(X)}$. One defines $dim(Conv(X)) := dim(L_X)$. Identify $L_X \approx \mathbb{R}^{dim(L_X)}$ (as topological spaces) and take the interior, $Int(Conv(X)) \subseteq L_X$. Prove: $\overline{Int(Conv(X))} = \overline{Conv(X)}$. (The closures are taken inside $\mathbb{R}^n$.) Why do we need to take the interior inside L_X and not inside $\mathbb{R}^n$?
(d) Define the boundary, $\partial Conv(X) := \overline{Conv(X)} \setminus Int(Conv(X))$. Let $pt \in Int(Conv(X))$. Suppose X is bounded. Prove: any ray through pt intersects $\partial Conv(X)$ in precisely one point.
(e) Suppose X is bounded. Prove: $Int(Conv(X)) \overset{homeo}{\approx} Ball_{dim(Conv(X))}$, $\overline{Conv(X)} \overset{homeo}{\approx} \overline{Ball_{dim(Conv(X))}}$, $\partial(Conv(X)) \overset{homeo}{\approx} S^{dim(Conv(X))-1}$.

(7) Points $\underline{a}_0, \ldots, \underline{a}_k \in \mathbb{R}^n$ are called "in general position" if the minimal plane that contains them is of dimension k. Prove:
(a) If the points $\{\underline{a}_i\}$ are in general position then any subset of them is also in general position.
(b) The points are in general position iff the system $\{\sum_{i=0}^k t_i = 0 = \sum_{i=0}^k t_i \underline{a}_i\}$ has only the trivial solution.

(8) (a) A k-simplex is the convex hull $\Delta_k := Conv(\underline{a}_0, \ldots, \underline{a}_k)$ for some points in general position. Prove that the boundary, $\partial \Delta_k$, is the union of $(k-1)$-simplexes. These are called $(k-1)$-faces of Δ_k. By taking their boundaries on gets $(k-2)$-faces, and so on. For every j denote the number of j-faces of Δ_k by d_j. Compute d_j. Compute $\sum (-1)^j d_j$.
(b) Split a cube in $\mathbb{R}^3$ into the union of 3-simplexes whose interiors are (pairwise) disjoint. What is the minimal possible number of such 3-simplexes? What about the box $[0,1]^n \subset \mathbb{R}^n$?

(9) (a) For any polygon in $\mathbb{R}^2$, with connected interior, prove: $\sharp(vertices) - \sharp(edges) + 1(face) = 1$.
(b) For any connected graph consider $\sharp(vertices) - \sharp(edges)$. Prove that this is 1 for a tree. For a planar graph (not a tree) prove: $\sharp(vertices) - \sharp(edges) = 2-$(the number of connected components of $\mathbb{R}^2 \setminus \Gamma$).
(c) Verify for as many polytopes in $\mathbb{R}^3$ as you can: $\sharp(vertices) - \sharp(edges) + \sharp(faces) - 1(body) = 1$.

(10) (a) Let $\{G_\alpha\}$ be an arbitrary collection of groups. What is the difference between $\prod G_\alpha$ and $\oplus G_\alpha$?
(b) Let $[G, G] := \{aba^{-1}b^{-1} | \ a, b \in G\}$. Prove: $[G, G] \lhd G$. Prove: $G/_{[G,G]}$ is an abelian group.
(c) Prove: every subgroup of a free abelian group is free. Is this subgroup necessarily a direct summand?
(d) Recall the Smith normal form over $\mathbb{Z}$ (e.g. look in wiki).
(e) Recall the structure theorem of finitely generated abelian groups.