Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 1

(1) (a) Prove: the quotient topology on $X/\sim$ is the maximal topology for which the projection $X \overset{pr}{\to} X/\sim$ is continuous.

(b) Prove: every finite connected graph can be obtained as a quotient space of $[0,1]$.

(c) Given X with two equivalences, $\overset{1}{\sim}, \overset{2}{\sim}$, prove: $X/(\overset{1}{\sim},\overset{2}{\sim}) \approx {}^{X/\overset{1}{\sim}}\Big/_{\overset{2}{\sim}/\overset{1}{\sim}}$.

(d) Let $D \subset X$ be a (closed) disc. Does there hold $X/D \overset{homeo}{\approx} X$? And what if $D \subset Ball \subset X$?

(2) (a) Identify the spaces. In the compact case specify a (possibly small) cell structure:
i. $S^2/\coprod S^1$ (n disjoint circles on S^2). ii. $\mathbb{R}^2/\text{a finite tree}$. iii. $S^n/\text{a finite subset}$.
iv. $S^1 \times S^1/S^1 \times pt$. v. $\mathbb{R}^n/\big(\underline{x} \sim \underline{y} \text{ if } \underline{x} - \underline{y} \in \mathbb{Z}^n\big)$. vi. $\mathbb{R}^2/\big((x_1,y_1) \sim (x_2,y_2) \text{ if } x_1 + y_1^2 = x_2 + y_2^2\big)$.

(b) Prove: $D^2/(x,y) \sim (-x,-y) \approx D^2/(x,y) \sim (x,-y) \approx D^2/(x,y) \sim (y,x)$

(c) Are the spaces $\mathbb{R}^2$, $\mathbb{R}^2/ray$ homeomorphic? (Here $ray \subset \mathbb{R}^2$ is e.g. $\{y = 0,\ x \geq 0\}$).

(3) (a) Present the Klein bottle as a quotient: i. of the Möbius strip, ii. of the cylinder,
iii. $S^1 \times S^1/(z,w) \sim (-z,\bar{w})$, where $S^1 \times S^1 = \{(z,w),\ |z| = 1 = |w|\} \subset \mathbb{C}^1 \times \mathbb{C}^1$

(b) Present $\mathbb{RP}^2$ as: i. a quotient of the Möbius strip, ii. a quotient of D^2,
iii. as the Möbius strip with a disc glued to the boundary.

(c) Prove: $\mathbb{CP}^1 \approx S^2$, $\mathbb{RP}^1 \approx S^1$, $\mathbb{RP}^2 \sharp \mathbb{RP}^2 \approx$ Klein bottle.

(d) Present the handle $(S^1 \times S^1 \setminus \overset{\circ}{D}{}^2)$ as a quotient of $[0,1]^2$. Using this present $\sharp^g(S^1 \times S^1)$ as a quotient of a planar polygon. Give a cell structure (small as possible). Construct a deformation retraction of $\sharp^g(S^1 \times S^1) \setminus \{pt\}$ to a one-dimensional cell complex with one cell e^0.

(e) Present the crosscap $(\mathbb{RP}^2 \setminus \overset{\circ}{D}{}^2)$ as a quotient of $[0,1]^2$. Present $\sharp^g(S^1 \times S^1)\ \sharp^g(\mathbb{RP}^2)$ as a quotient of a planar polygon. Give a cell structure (small as possible). Construct a deformation retraction of $\sharp^g(S^1 \times S^1) \setminus \{pt\}$ to a one-dimensional cell complex with one cell e^0.

(4) (a) Obtain the standard/canonical cell decomposition for $\mathbb{RP}^n$, $\mathbb{CP}^n$.

(b) For $\Bbbk = \mathbb{R}, \mathbb{C}$ define $\Bbbk\mathbb{P}^\infty$ as the space of lines in $\Bbbk^\infty = \cup \Bbbk^n$. What is the cell structure?

(c) Give infinitely many examples of non-homeomorphic spaces whose cell decomposition is e^2, e^1, e^0.

(d) For any X prove: $X^n/X^{n-1} \approx \vee_\alpha S^n_\alpha$, with one sphere for each n-cell of X^n

(5) (a) Prove that homotopy equivalence (of maps/spaces) is an equivalence relation.

(b) Construct explicit homotopy from an arbitrary map $[0,1] \overset{f}{\to} \mathbb{R}$ satisfying $f(0) = 0,\ f(1) = 1$ to the identity map, $Id(x) = x$.

(c) A subset $X \subset \mathbb{R}^n$ is called a "star-set" if for some point $\underline{x}_0 \in X$ any segment from $\underline{x}_0$ to any other point of X lies inside X.

(i) Prove: any star subset of $\mathbb{R}^n$ is contractible.

(ii) Prove that being contractible is a topological property (i.e. if X, Y are homeomorphic and X is contractible then Y as well).

(iii) Is it true that any two maps from a star set, $X \overset{f,f'}{\to} Y$ are homotopic? (What is the simple additional condition?)

(d) Suppose X retracts to A_X while Y to A_Y. What can be said about $X \times Y$, $X \vee Y$, $Cone(X)$, SX?

(e) Suppose the maps $X \overset{f,f'}{\to} Y$ are homotopic and the maps $Y \overset{g,g'}{\to} Z$ are homotopic. Construct an explicit homotopy of the maps $g \circ f$, $g' \circ f'$.

(f) Prove: the maps $X \overset{f,f'}{\to} Y_1 \times Y_2$ are homotopic iff $p_1(f) \sim p_1(f')$ and $p_2(f) \sim p_2(f')$.