Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 2

The questions for submission: 1b,1c,2c,2d,4a,4b,7c. The deadline: 11.11

(1) (a) Construct a deformation retration of $S^1 \times S^1 \setminus (n\ points)$ to a 'basis' circle with n other circles attached to it (at distinct points). Check that the later space is homotopic to $\vee^{n+1} S^1$.

(b) Suppose n circles in $\mathbb{R}^3$ are 'ball-separated' (i.e. each of them lies in a standard ball that does not intersect any other circle). Show that the complement, $\mathbb{R}^3 \setminus (n\ circles)$, deformation retracts to $\vee^n (S^2 \vee S^1)$.

(c) Construct a deformation retraction $\Bbbk\mathbb{P}^n \setminus \{pt\} \to \Bbbk\mathbb{P}^{n-1}$, for $\Bbbk \in \mathbb{R}, \mathbb{C}$, $n \geq 2$.

(d) Compute $\pi_1(\mathbb{RP}^n)$, $\pi_1(\mathbb{CP}^n)$. (In the same way as we computed $\pi_1(S^n)$, $n > 1$.)

(e) Questions 1, 2, 9, 10, 21 on pg. 19 of [Hatcher].

(2) (a) Prove that path homotopy is an equivalence relation. (Construct explicit homotopies!)

(b) Prove that the product of homotopy types of paths, $[f] \cdot [g] = [f \circ g]$, is well defined (i.e. does not depend on the representatives).

(c) Given the paths $[0,1] \stackrel{f,g,h}{\to} X$ such that $f(1) = g(0)$ and $g(1) = h(0)$ prove that $f \circ (g \circ h) \sim (f \circ g) \circ h$. (Note that $f \circ (g \circ h) \neq (f \circ g) \circ h$.)

(d) Given a path f from x_0 to x_1 define $f^{-1}(s) = f(1-s)$. Prove: $f \circ f^{-1} \sim e_{x_0}$ and $f^{-1} \circ f \sim e_{x_1}$. (Here e_* is a constant path.)

(3) (a) For any X show that $Cone(X)$ deformation retracts to a point.
Suppose X has the cell structure $\{m_i e^i\}$, with $m_i \geq 0$. What is the cell structure of $Cone(X)$

(b) The suspension of a space X is defined as $SX := {}^{X \times [0,1]}\!/\!\big(X \times \{0\}, \quad X \times \{1\}\big)$. Identify the space SS^n.

 (i) Express SX as two cones with a common basis.

 (ii) Suppose X has the cell structure $\{m_i e^i\}$ (with $m_i \geq 0$). What is the cell structure of SX.

 (iii) Let Γ be a connected graph. Prove: $S\Gamma$ deformation retracts to $\vee^n S^2$. What is n in terms of Γ?

(c) The join of spaces X, Y is defined as the space of all the segments between the points of X, Y:

$$X * Y := \frac{X \times Y \times [0,1]}{(x, y_1, 0) \sim (x, y_2, 0), \quad (x_1, y, 1) \sim (x_2, y, 1)}.$$

 (i) Suppose the cell structure of X is $\{m_i e^i\}$, and that of Y is $\{n_j e^j\}$. What is the cell structure of $X * Y$?

 (ii) Questions 18,19,20 on page 19 of [Hatcher].

(4) (a) Show that a map $\mathbb{R}^2 \supseteq S^1 \stackrel{f}{\to} Y$ extends continuously to the whole $\mathbb{R}^2$ iff the induced homomorphism $\pi_1(S^1) \stackrel{f_*}{\to} \pi_1(Y)$ is trivial.

(b) Prove: for any two polynomials of the same degree, $p(x), q(x) \in \mathbb{C}[x]$ there exists $r > 0$ such that for any $R > r$ the maps $\{z|\ |z| = R\} \stackrel{p,q}{\to} \mathbb{C} \setminus \{0\}$ are homotopic.

(5) Suppose a map $S^n \stackrel{f}{\to} S^n$ has no fixed points, i.e. $f(x) \neq x$. Prove that f is homotopic to $S^n \ni x \to (-x) \in S^n$.

(6) Questions 5, 10, 16, 18 on pg 38-39 of Ha

(7) (a) Fix the matrix $A = \begin{bmatrix} n_1 & m_1 \\ n_2 & m_2 \end{bmatrix} \in Mat_{2\times 2}(\mathbb{Z})$ and consider the map $S^1 \times S^1 \stackrel{\phi_A}{\to} S^1 \times S^1$, $\phi_A(z,w) = (z^{n_1} w^{m_1}, z^{n_2} w^{m_2})$, for some fixed $n_i, m_i \in \mathbb{Z}$. Prove that ϕ_A is homeomorphism iff $A \in GL(2, \mathbb{Z})$, i.e. $det(A) = \pm 1$.

(b) Fix the standard basis of $\mathbb{Z}^2$. Draw the loops on the torus whose classes are $(1,0)$, $(0,1)$, $(1,1)$.

(c) Compute the presentation matrix of the map $\pi_1(S^1 \times S^1) \stackrel{(\phi_A)_*}{\to} \pi_1(S^1 \times S^1)$.

(d) Prove that there exists a self-homeomorphism of the torus that sends a loop of the class $(3,4)$ to the loop of the class $(1,0)$. Could you visualize it?