Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 3

The questions for submission: 2a, 3b, 3c, 4b, 5d, 5e, 11. The deadline: 26.11

(1) (a) Let X be a cell complex and $Z \subset X$ a compact subspace. Prove/disprove:
  (i) Z is contained in a finite number of cells.
  (ii) Z intersects a finite number of cells.
(b) Prove: any finite cell complex embeds into some $\mathbb{R}^n$.

(2) (a) We have proved: if X is path connected then $\pi_1(X, x_0) \xrightarrow{\phi} \pi_1(X, x_1)$. The isomorphism ϕ is not canonical and in general depends on the choice of the path $x_0 \rightsquigarrow x_1$. Prove: ϕ does not depend on the choice of the path iff $\pi_1(X, x_0)$ is abelian.
(b) Given a continuous map of pointed spaces, $(X, x_0) \xrightarrow{f} (Y, y_0)$, and a path $x_0 \overset{\delta}{\rightsquigarrow} x_1$, construct the commutative diagram:
What are the maps δ_*, $(f \circ \delta)_*$?

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,y_0) \\ \delta_* \downarrow & & \downarrow (f\circ\delta)_* \\ \pi_1(X,x_1) & \xrightarrow{f_*} & \pi_1(Y,y_1) \end{array}$$

(3) (a) Questions 1, 10, 16(f), 17 in [Hatcher], pg 38-39.
(b) Denote by $X_{g,q}$ the sphere with g handles and q-crosscaps. Compute $\pi_1(X_{g,q})$. Prove: if $X_{g,q} \sim X_{g',q'}$ then $2g + q = 2g' + q'$ and either $q = 0 = q'$ or $q, q' > 0$.
(c) Questions 2, 8, 9, in [Hatcher], pg 52-55

(4) (a) Compute $\pi_1(\mathbb{R}^3 \setminus X)$ when X is an ordinary circle, two linked circles, two unlinked circles. (See example 1.23 in [Hatcher].)
(b) Let X be the union of k lines through the origin in $\mathbb{R}^n$, $n \geq 3$. Compute $\pi_1(\mathbb{R}^n \setminus X)$.

(5) (a) Classify all the coverings $\tilde{X} \xrightarrow{p} X$ with $\tilde{X}$ path connected and X simply connected.
(b) Which of the following spaces can be covered by $S^1 \times \cdots \times S^1$: S^n, $\mathbb{RP}^n$, $\mathbb{RP}^n \vee \mathbb{RP}^n$, $\mathbb{CP}^n$.
(c) Construct the covering maps: i. From $\mathbb{R}^2$ to the torus. ii. From the sphere with $g \geq 2$ handles to the sphere with 2 handles. iii. From the cylinder to the Möbius band. iv. From the torus to the Klein bottle.
What are the possible degrees of covering in each case? (i.e. the number of preimages of a point)
(d) Let α, β by *any* loops generating $\pi_1(S^1 \times S^1)$. Prove that they intersect.
(e) Prove: $\pi_1(GL(n, \mathbb{C})) \supseteq \mathbb{Z}$. What about $SL(n, \mathbb{C})$, $U(n)$, $SU(n)$?

Review questions

(6) Identify the space of monic complex quadratic polynomials, $z^2 + az + b$, with $\mathbb{C}^2$. Let $X \subset \mathbb{C}^2$ be the subset of polynomials with distinct roots. Prove: $X \sim S^1$.

(7) Let X= sphere with g handles. Prove that $X \setminus \{pt\}$ deformation retracts to a wedge of circles. (How many?)

(8) Use the standard cell structure of S^n to embed $S^n \vee S^m \subset S^n \times S^m$. Prove:
i. $S^m \times S^n / S^n \vee S^m \approx S^{n+m}$. (contraction) ii. Prove: $S^m \times S^n \setminus \{point\} \sim S^n \vee S^m$.

(9) Let $S^\infty = \{\sum x_i^2 = 1\} \subset \mathbb{R}^\infty$. (Recall: $\mathbb{R}^\infty = \cup \mathbb{R}^n$.) Prove that S^∞ is contractible. (Hint: prove that the identity map is homotopically equivalent to the map $(x_1, x_2, \dots) \to (0, x_1, x_2, \dots, ..)$.)

(10) Consider the groups $SU(2) \subset Mat_{2\times 2}(\mathbb{C})$, $SO(3) \subset Mat_{3\times 3}(\mathbb{R})$ as topological subspaces. Prove: $SU(2) \approx S^3$ and $SO(3) \approx \mathbb{RP}^3$. (For the later you can look at the bottom of pg. 293 of [Hatcher].)

(11) Construct explicit deformation retracts: $GL(n, \mathbb{R}) \sim O(n)$, $GL(n, \mathbb{C}) \sim U(n)$. (Hint: polar decomposition.)