Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 4

The questions for submission: 1b, 1d, 2a, 2c, 3d, 4c, 4d. The deadline: 9.12

(1) (a) Construct the universal covering of $S^1 \vee S^1$, $\square\square$ (one dimensional cell complex).

(b) Construct/classify all the path-connected coverings of $\mathbb{RP}^n \vee \mathbb{RP}^m$. (It is worth to begin with the coverings corresponding to the subgroups: $< abab\cdots a >$, $< abab\cdots b >$, $< ababa, babab >$ of $\pi_1(\mathbb{RP}^n \vee \mathbb{RP}^m)$.)

(c) Suppose the universal coverings $\tilde{X} \stackrel{p_X}{\to} X$, $\tilde{Y} \stackrel{p_Y}{\to} Y$, are of degrees $1 < n_X, n_Y < \infty$. Construct the universal covering of: i. $Cone(X)$ ii. $X \times Y$ iii. $X \vee Y$.

(d) We work in the category of path-connected, locally path-connected, semi-locally simply connected, pointed topological spaces. Prove that any continuous map $(X, x_0) \stackrel{\phi}{\to} (Y, y_0)$ lifts to that of the univesal coverings, as on the diagram.

$$\begin{array}{ccc} (\tilde{X}, \tilde{x}_0) & \stackrel{\tilde{\phi}}{\dashrightarrow} & (\tilde{Y}, \tilde{y}_0) \\ \downarrow & & \downarrow \\ (X, x_0) & \stackrel{\phi}{\to} & (Y, y_0) \end{array}$$

Moreover, this lifting satisfies: $\tilde{\phi} \circ \tilde{\psi} = \widetilde{\phi\psi}$ and $\widetilde{Id} = Id$. Thus the universal covering is a functor.

(2) (a) Give a presentation of a space via G-covering, $X = \mathbb{R}^2/G$, in the following cases: i. $X = S^1 \times S^1$. ii. $X =$Möbius band. iii. $X =$Klein bottle.

(b) Compute $Aut(p)$ for a 2:1 covering $\tilde{X} \stackrel{p}{\to} X$.

(c) Take the group $\mu_n = \{\xi \in \mathbb{C} |\ \xi^n = 1\}$ and consider the action $\mu_n \circlearrowright S^{2m-1} \subset \mathbb{C}^m$, by $(z_1, \ldots, z_m) \to (\xi^{l_1} z_1, \ldots, \xi^{l_m} z_m)$, for some inegers satisfying $gcd(l_i, n) = 1$. Check that this is an "admissible" action and compute $\pi_1(S^{2m-1}/\mu_n)$. (This quotient is called the *Lens* space, $L(n, l_1, \ldots, l_m)$.)

(d) Given a covering $\tilde{X} \stackrel{p}{\to} X$ of degree n and $x_0 \in X$, the restriction $Aut(p) \to Aut(p)|_{p^{-1}(x_0)}$ defines the morphism $Aut(p) \to \Sigma_n$ (permutation group on n elements). Prove that this is injection, i.e. any automorphism of the covering is fully determined by its action on a(ny) fibre. When is this surjection?

(e) A section of a covering $\tilde{X} \stackrel{p}{\to} X$ is a continuous map $X \stackrel{s}{\to} \tilde{X}$ satisfying $p \circ s = Id$. Prove: a path-connected G-covering admits a section iff $G = \{e\}$.

(f) Prove that the definition of the normal covering does not depend on the base point. (i.e. if the action $Aut(p) \circlearrowright p^{-1}(x_0)$ is transitive for some x_0 then it is transitive for any other $x \in X$)

(g) q. 4, 9, 18, 27, 32 of [Hatcher], pg79-82.

(3) Leftovers from the class.

(a) For the homotopy lifting property we proved: if $\tilde{X} \stackrel{p}{\to} X$ is a covering and $Y \stackrel{f_0}{\to} X$ lifts to $Y \stackrel{\tilde{f}_0}{\to} \tilde{X}$ then any homotopy of f_0 lifts to a (set-theoretic) map $Y \times [0,1] \stackrel{\{\tilde{f}_t\}}{\to} X$. Prove that this lifted map is continuous.

(b) For the lifting criterion we have assumed $(Y, y_0) \stackrel{f}{\to} (X, x_0)$ and $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$, and constructed the set-theoretic map $(Y, y_0) \stackrel{\tilde{f}}{\to} (\tilde{X}, \tilde{x}_0)$. Prove that $\tilde{f}$ is continuous.

(c) Go over all the details in the construction of the universal cover. Fill in all the parts that were left on the lecture.

(d) Prove: $[\gamma] \in \pi_1(X, x_0)$ induces an element of $Aut(p)$ iff $[\gamma] \in N(p_*\pi_1(\tilde{X}, \tilde{x}_0))$.

Review questions

(4) (a) Identify/describe the topological spaces:

(i) $Mat_{m\times n}(\mathbb{R})/GL(m) \times GL(n)$, with the action $A \to UAV^{-1}$.

(ii) $Mat^{sym}_{n\times n}(\mathbb{R})/GL(n)$, $Mat^{skew-sym}_{n\times n}(\mathbb{R})/GL(n)$, with the action $A \to UAU^t$.

(iii) $Mat_{n\times n}(\mathbb{C})/GL(n,\mathbb{C})$, with the action $A \to UAU^{-1}$.

(b) Construct deformation retractions $S^3 \setminus S^1 \to S^1$, $S^n \setminus S^m \to S^{n-m-1}$, $\mathbb{RP}^n \setminus \mathbb{RP}^m \to \mathbb{RP}^{n-m-1}$.

(c) Compute $\pi_1(X \setminus \{n - punctures\})$ when X is: i. $S^2 \sharp^g (S^1 \times S^1)$ ii. $S^2 \sharp^q \mathbb{RP}^2$

(d) Suppose X is path-connected. Compute $\pi_1(S(X))$ (suspension).

(e) Let $\tilde{X} \stackrel{p}{\to} X$ be a covering map of compact topological spaces. Demonstrate a cell structure $\{e^{n_\alpha}_\alpha\}$ on X for which $\{p^{-1}(e^{n_\alpha}_\alpha)\}$ is a cell structure on $\tilde{X}$.