Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 5

The questions for submission: 1, 3b, 3d, 3f, 5b, 5a, 5b, 6a . The deadline: 19.12

All the homologies are taken with $\mathbb{Z}$-coefficients. Except for q.1 H_* denotes the singular homology. All the spaces are locally contractible at all their points.

(1) Compute the simplicial homology, $H_*(S^n)$, $H_*(\mathbb{RP}^n)$, by fixing some (nice) Δ-complex structures.

(2)
- (a) q. 14,15 on page 132 of [Hatcher]
- (b) (Splitting lemma) Given a short exact sequence of abelian groups, $0 \to A \stackrel{i}{\to} B \stackrel{j}{\to} C \to 0$, prove that the following are equivalent:
  - (i) (i admits a section) There exists a morphism $B \stackrel{s_i}{\to} A$ satisfying $s_i \circ i = Id_A$.
  - (ii) (j admits a section) There exists a morphism $C \stackrel{s_j}{\to} B$ satisfying $j \circ s_j = Id_C$.
  - (iii) There exists an isomorphism $B \stackrel{\phi}{\longrightarrow} A \oplus C$ inducing the commutative diagram:

$$\begin{array}{ccccc} 0 \to A & \stackrel{i}{\to} & B & \stackrel{j}{\to} & C \to 0 \\ || & & \downarrow \phi & & || \\ 0 \to A & \stackrel{Id \oplus 0}{\to} & A \oplus C & \stackrel{0 \oplus Id}{\to} & C \to 0 \end{array}$$

- (c) Prove: if $Y \subset X$ is a retract then $\{H_n(X) = H_n(Y) \oplus G_n\}_n$, for some abelian groups $\{G_n\}$.
- (d) Prove: an abelian group C is free iff every exact sequence $0 \to A \stackrel{i}{\to} B \stackrel{j}{\to} C \to 0$ splits.

(3)
- (a) Compute $H_*(S^n)$, $H_*(\mathbb{CP}^n)$, $H_*(\mathbb{RP}^n)$ via the homology of pair and excision. (In as many ways as possible.)
- (b) Let X be connected and $Y \subset X$ a finite set of points. Express $H_*({}^X\!/_Y)$ via $H_*(X)$ and $|Y|$.
- (c) q. 20, 22 on page 132 of [Hatcher]
- (d) Prove: $H_*(\vee X_\alpha) \approx \oplus H_*(X_\alpha)$. (Here each X_α is locally contractible at the vertex of the wedge.)
- (e) Compute $H_*(X)$ for a connected finite graph, in terms of the cardinality of $X \setminus Y$, where Y is a maximal subtree.
- (f) Prove: $H_j(S^1 \times S^1) \approx H_j(S^1 \vee S^1 \vee S^2)$ for any $j \geq 0$. What about π_1?

(4)
- (a) Fill all the details in the proof of the zig-zag lemma.
- (b) Prove Brouwer's theorem: any continuous map $D^n \stackrel{f}{\to} D^n$ has a stable point.

(5)
- (a) Suppose a continuous map $\mathbb{P}^n_\mathbb{C} \stackrel{f}{\to} \mathbb{P}^n_\mathbb{C}$, induces non-zero map on the top homology. Prove: f is surjective.
- (b) Does there exist a retraction of a Möbius strip onto its boundary?

(6) **Review questions**
- (a) Given a map $S^1 \times S^1 \stackrel{f}{\to} D^2$, suppose the restriction $f^{-1}(\partial D^2) \stackrel{f|}{\to} \partial(D^2)$ induces a nonzero map of the fundamental groups. Does f have to be surjective?
- (b) For any $k < n$ construct explicit deformation retracts: $\mathbb{P}^n_\mathbb{R} \setminus \{x_1, \dots, x_k\} \rightsquigarrow \mathbb{P}^{n-k}_\mathbb{R}$, $\mathbb{P}^n_\mathbb{C} \setminus \{x_1, \dots, x_k\} \rightsquigarrow \mathbb{P}^{n-k}_\mathbb{C}$. What happens for $k \geq n$?
- (c) Let $G \circlearrowright X$ be a free action of a finite group on a Hausdorff space. Prove: this action induces the covering map $X \stackrel{p}{\to} {}^X\!/_G$.
- (d) Consider the group action $G \circlearrowright \mathbb{R}^2$ generated by the transformations $(x, y) \stackrel{a}{\to} (x+1, 1-y)$, $(x, y) \stackrel{b}{\to} (1-x, y+1)$. Is ${}^{\mathbb{R}^2}\!/_G \approx \mathbb{RP}^2$? Is the projection $\mathbb{R}^2 \stackrel{q}{\to} {}^{\mathbb{R}^2}\!/_G$ a covering map?