Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 6

The questions for submission: 1d, 1e, 1f, 2b, 2d, 3b, 4b. The deadline: 6.01.2019

All the homologies are taken with $\mathbb{Z}$-coefficients. All the spaces are locally contractible at all their points.

(1) Leftovers from the class and the Mayer-Vietoris sequence. Let $A, B \subset X$ such that $Int(A) \cup Int(B) = X$.
   (a) Prove: for any singular simplex $\Delta^n \xrightarrow{\sigma} X$ there exists a finite partition into simplices, $\Delta^n = \cup \Delta^n_i$, such that $Int(\Delta^n_i) \cap Int(\Delta^n_j) = \varnothing$ and for any i holds: $\sigma(\Delta^n_i) \subset A$ or $\sigma(\Delta^n_i) \subset B$.
   (b) Prove: for this partition $\Delta^n = \cup \Delta^n_i$ and $\sigma_i := \sigma|_{\Delta^n_i}$ holds: $\sigma - \sum \sigma_i \in B_n(X)$.
   (c) Establish the exact sequence of complexes $0 \to C_\bullet(A \cap B) \xrightarrow{\phi} C_\bullet(A) \oplus C_n(B) \xrightarrow{\psi} C_\bullet(A + B) \to 0$, where $\psi(x, y) = x + y$.
   (d) Apply the zig-zag lemma (and notice that $H_n(A + B) = H_n(X)$) to get the exact sequence of Mayer-Vietoris: $\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0$.
   (e) Suppose $X \subset \mathbb{R}^n$ admits an open neighborhood, $\mathcal{U}$, that deformation retracts onto X. Prove: $H_j(\partial\mathcal{U} \setminus X) \approx H_j(X) \oplus H_j(\mathbb{R}^n \setminus X)$. Compute $H_*(\mathbb{R}^n \setminus X)$, $n \geq 3$, when X is a finite disjoint union of circles. (In particular this does not depend on whether the circles are linked and/or knotted.)
   (f) Fix a topological space X and glue N copies of $Cone(X)$ by identifying the bases, $X_i \times \{0\} \approx X_j \times \{0\}$. Relate/express $H_*(\coprod Cone(X)/\sim)$ via $H_*(X)$. Do this in as many ways as possible, e.g. via splitting into nice subspaces/contracting special subspaces/puncturing some points. (Compare to q. 29 on page 158 of [Hatcher].)
   (g) Prove the 5-lemma.

(2) Degrees of maps $S^n \to S^n$
   (a) Check: $deg(Id) = 1$; $deg(f \circ g) = deg(f) \cdot deg(g)$; $deg(f)$ is a homotopy invariant.
   (In particular: if f is non-surjective then $deg(f) = 0$; if f is a homeomorphism then $deg(f) = \pm 1$.)
   (b) Define the reflection $S^n \xrightarrow{f} S^n$ by $f(x_0, \ldots, x_n) = (-x_0, x_1, \ldots, x_n)$. Prove: $deg(f) = -1$.
   (Cover S^n by two simplices, $\Delta^n_\pm = \{x_0 \gtrless 0\}$, and check the map $f_* \circlearrowleft H^\Delta_n(S^n)$.)
   (c) Let $f(\underline{x}) = -\underline{x}$. Prove: $deg(f) = (-1)^{n+1}$.
   (d) Fix k non-intersecting subspheres, $\coprod S^{n-1}_j \subset S^n$ (e.g. $\{x_0 = \frac{j}{k}\} \subset S^n$, for $j = 0 \ldots k-1$). Let f be the composition of maps $S^n \to S^n/\coprod_j S^{n-1} \approx \vee^{k+1} S^n_i \xrightarrow{\vee^{k+1}\phi_i} S^n$. Here $\{\phi_i\}$ are some homeomorphisms. Prove: $H_n(S^n) \xrightarrow{f_*} H_n(S^n)$ acts by $1 \to (\pm 1, \ldots, \pm 1) \cdot (1, \ldots, 1)^t = \sum(\pm 1)$. By composing ϕ with inversions of $\{S^n_i\}$ (if needed) we can achieve: $deg(f) = k + 1$. (Thus giving a map of any prescribed degree.)

(3) (a) Questions 20, 21, 22, 23 on page 157 of [Hatcher].
   (b) Consider the complex algebraic curve, $C_d := \{[x_0 : x_1 : x_2] |\ x_0^d + x_1^d + x_2^d = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$. Prove that this is a smooth compact manifold of real dimension 2. One can prove that this manifold is orientable. Therefore $C \overset{homeo}{\approx} \sharp^g(S^1 \times S^1)$. Consider the projection $C \to \{[x_1 : x_2]\} = \mathbb{P}^1_{\mathbb{C}} = \{x_0 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$. Verify that this is a covering, except for a finite number of points. (Which points?) Compute $\chi(C)$. Prove: $g = \binom{d-1}{2}$.

(4) (a) q.12 on page 156 of [Hatcher].
   (b) Suppose a (continuous) map $\mathbb{P}^n_{\mathbb{R}} \xrightarrow{f} \mathbb{P}^n_{\mathbb{R}}$, $n > 1$ induces an injection $\pi_1(\mathbb{P}^n_{\mathbb{R}}, x_0) \xrightarrow{f_*} \pi_1(\mathbb{P}^n_{\mathbb{R}}, x_0)$. Prove that f lifts to the map $S^n \xrightarrow{\tilde{f}} S^n$ and $\tilde{f}(-x) = -\tilde{f}(x)$.