Basic Concepts in Topology and Geometry

201.2.522. Fall 2018 (D.Kerner)

Homework 7

All hte groups are abelian. Being finitely generated is abbreviated to f.g.
(Co)Homology with integer coefficients is denoted by $H_*(X)$, $H^*(X)$.

(1) (a) Prove that $Hom(-,-)$ is a covariant functor in the first variable and a contravariant functor in the second.
(b) Prove: $Hom(\oplus A_\alpha, G) \approx \prod Hom(A_\alpha, G)$. In particular, while the chains $\{C_n(X;G)\}$ consist of finite linear combinations, the co-chains $\{C^n(X;G)\}$ involve arbitrary linear combinations.
(c) Suppose A, B are f.g. free abelian groups or vector spaces, with a morphism $A \stackrel{\phi}{\to} B$. Fix some bases $\{a_i\}$, $\{b_j\}$, and the dual bases $\{a_i^*\}$, $\{b_j^*\}$. Prove: $[\phi^*] = [\phi]^t$. (Why does one need the f.g. assumption?) Conclude, that in this case $(\phi^*)^* = \phi$.
(d) Given a morphism of (not necessarily f.g.) abelian groups $A_1 \oplus A_2 \stackrel{\{\phi_{ij}\}}{\to} B_1 \oplus B_2$ and its dual $B_1^* \oplus B_2^* \stackrel{\{(\phi^*)_{ij}\}}{\to} A_1^* \oplus A_2^*$, prove: $(\phi^*)_{ij} = (\phi_{ji})^*$.
(e) Is the sequence $0 \to Hom(\mathbb{Z}/_{(n)}, G) \to G \stackrel{\times n}{\to} G \to Ext(\mathbb{Z}/_{(n)}, G) \to 0$ exact for any G?
(f) Suppose A is f.g. and torsion free. Prove: $Ext(A, G) = 0$ for any G.

(2) (a) Suppose A (not necessarily f.g.) is torsion free. Prove: $Tor(A, G) = 0$, for any G.
(b) Is the sequence $0 \to Tor(\mathbb{Z}/_{(n)}, A) \to A \stackrel{\times n}{\to} A \to A \otimes Z/_{(n)} \approx A/_{nA} \to 0$ exact for any A?
(c) Prove: if A, B are f.g. then $Tor(A, B) \approx Torsion(A) \otimes Torsion(B)$.

(3) (a) Prove: $H^0(X; G) = \prod G$ (the product over the path-connected components of X).
(b) Question 5 on page 205 of Hatcher.

(4) (a) We have proved in the class: $H^n(X;G) = Hom(H_n(X), G) \oplus Ext(H_{n-1}(X), G)$ and $H_n(X;G) = (H_n(X) \otimes G) \oplus Tor(H_{n-1}(X), G)$. Go over all the details of the proof.
(b) Prove: $H_n(X;G) \approx Hom(H^n(X), G) \oplus Ext(H^{n+1}(X), G)$ and $H^n(X;G) \approx (H^n(X) \otimes G) \oplus Tor(H^{n+1}(X), G)$.
(c) Compute the groups $H^*(\sharp^g(S^1 \times S^1)\sharp^q \mathbb{RP}^2, \mathbb{Z}/_{(17)})$, $H^*(\mathbb{RP}^n \vee \mathbb{RP}^m, \mathbb{Z}/_{(3)} \oplus \mathbb{Z}/_{(2)})$, $H^*(\mathbb{RP}^n \times \mathbb{CP}^m, \mathbb{Q})$.

(5) (a) We have computed the ring $H^*(\sharp^g(S^1 \times S^1); \mathbb{Z})$. Go over all the details.
(b) Compute the ring $H^*(\sharp^q \mathbb{RP}^2; \mathbb{Z})$.
(c) Choose a particular cell structures for $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$, for which the cellular chain complexes are isomorphic. Prove that the cohomology rings of $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ are non-isomorphic.
(d) Question 18 on page 230 of Hatcher.

(6) (a) Prove that connectedness and path-connectedness coincide for topological manifolds.
(Hint: let $\mathcal{U}_x \subset M$ be the set of all points path-connected to a given x.)
(b) Prove: every non-orientable connected topological manifold admits a 2:1 covering by a connected orientable topological manifold.
(c) Questions 2, 3 on page 257 of Hatcher.
(d) Check the (non)orientability of Klein bottle, $\mathbb{RP}^n$, $\mathbb{CP}^n$.
(e) Prove: $X \times Y$ is orientable iff each of X, Y is.