# Introduction to Algebraic Curves

## and Algebraic Geometry

## Homework 0

Summer 2019 (D.Kerner)

Below $\mathbb{k}$ is one of $\mathbb{R}$, $\mathbb{C}$.

(1) Draw/imagine the following subsets of $\mathbb{k}^n$. At which points the subsets are smooth? (real/complex manifolds?) What are the singular points? How do the subsets look like near their singular points? Which subsets consist of "several components"? (i.e. are reducible) What is the(real/complex) dimension of the components? Do the subsets "separate" the ambient $\mathbb{k}^n$ into several components? Are the subsets/their complements simply connected?

(Some of these questions are not precise, as we did not define anything yet.)

i. $\{x^2 = y^2\} \subset \mathbb{k}^2$ ii. $\{y^2 = x^3\} \subset \mathbb{k}^2$ iii. $\{y^p = x^p\} \subset \mathbb{k}^2$ iv. $\{y^p = x^p\} \subset \mathbb{k}^3$

v. $\{xyz = 0\} \subset \mathbb{k}^3$ vi. $\{xz = 0 = yz\} \subset \mathbb{k}^3$ vii. $\{y^2 = x^2 + x^3\} \subset \mathbb{k}^2$ viii. $\{y^2 = x^2 z\} \subset \mathbb{k}^3$

Here some sets are singular. Can you realize them as a result of an "unfortunate" projection of some smooth sets? (e.g. $\{x^2 = y^2\}$ is the unfortunate projection of two disjoint lines in $\mathbb{k}^3$)

An **affine plane algebraic curve** is the subset $C := \{f(x,y) = 0\} \subset \mathbb{k}^2$, where $f(x,y) \in \mathbb{k}[x,y]$. The degree of the curve is defined as the degree of the polynomial $f(x,y)$, the largest among the total degrees of monomials participating in $f(x,y)$.

(2) (a) Define the group action of $\mathbb{k}^* := \mathbb{k} \setminus \{0\} \circlearrowright \mathbb{k}[x,y]$, by $f \to \lambda \cdot p$. Check that this gives a trivial action on curves.
(b) Consider the group actions $GL(2,\mathbb{k}) \circlearrowright \mathbb{k}^2$ (linear transformations) and $\mathbb{k}^2 \circlearrowright \mathbb{k}^2$ (parallel translations). Prove that the group generated by these is a semi-direct product, $\mathbb{k}^2 \rtimes GL(2,\mathbb{k})$.
(c) Write down the action of the group $\mathbb{k}^* \times \mathbb{k}^2 \rtimes GL(2,\mathbb{k})$ on the set of algebraic curves. (What this action does to the equation $\{f(x,y) = 0\}$?) Check that this action preserves the degree of curves.

(3) (Plane conics, i.e. curves of degree=2)
(a) Let $f(x,y) \in \mathbb{R}[x,y]$ be of degree 2. Prove that by the action of $\mathbb{k}^* \times \mathbb{k}^2 \rtimes GL(2,\mathbb{k})$ the curve $\{f(x,y) = 0\} \subset \mathbb{R}^2$ can be brought to one (and only one) of the following forms: $y^2 \pm x^2 = 0$, $y^2 \pm x^2 = 1$, $x^2 + y^2 = -1$, $y = x^2$, $x^2 = 1$, $x^2 = 0$. Draw the corresponding curves. (These are called: canonical forms of plane conics.) Which of the curves are smooth? (as sets)
(b) Let $f(x,y) \in \mathbb{C}[x,y]$ be of degree 2. What are the canonical forms in this case?
(c) Consider the rings $\mathbb{k}[x,y]/(x^2 - y^2)$, $\mathbb{k}[x,y]/(x^2 - y^2 + 1)$, $\mathbb{k}[x,y]/(x^2 - 1)$, $\mathbb{k}[x,y]/(x^2)$. Which of them are integral domains? Normal rings? Rings without nilpotents? PID?

(4) (Plane cubics) Trace the change of the curve $\{y^2 = x^3 + x^2 + \epsilon\} \subset \mathbb{R}^2$ for $\epsilon \in (-1,1)$. What happens at $\epsilon = 0$? (You can use the computer. Google: "plane cubic", "nodal cubic")

Trace the change of the curve $\{y^2 = x^3 + \epsilon \cdot x^2\} \subset \mathbb{R}^2$ for $\epsilon \in (-1,1)$.

(5) (a) Let $C = \{x^2 + y^2 = 1\} \subset \mathbb{k}^2$. Let $C \stackrel{\pi_x}{\to} \mathbb{k}^1$ be the projection onto the $\hat{x}$-axis. Describe the image, $\pi_x(C)$. How many preimages has a point $x \in \pi_x(C)$?
(b) Let $C = \{f(x,y) = 0\} \subset \mathbb{C}^2$ for some polynomial $f(x,y) \in \mathbb{C}[x,y]$, of degree $d$. Prove that the image of the projection $\pi_x$ is either a finite set of points or the $\hat{x}$-axis with a finite set of points removed. (e.g. $y(x^2 - 1) = 1$.) Suppose for every point $x$ of the $\hat{x}$-axis the number of preimages (in $C$) is finite. Prove that this number is generically $d$. (i.e. for all values of $x$, except for a finite subset, this number equal $d$.)

A **projective plane algebraic curve** is the subset $C := \{f(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2$. Here $[x_0 : x_1 : x_2]$ are the homogeneous coordinates, $f$ is a homogeneous polynomial. $deg(C) := deg(f)$.

(6) (a) Prove that the group action $\mathbb{k}^2 \rtimes GL(2,\mathbb{k})$ extends to an action on the set of projective plane curves. (Establish the embedding $\mathbb{k}^2 \rtimes GL(2,\mathbb{k}) \subset \mathbb{P}GL(3,\mathbb{k})$.)
(b) Write down canonical forms of projective conics for $\mathbb{k} = \mathbb{C}$.
(c) Prove that every two lines intersect. (No lines are parallel)
(d) Prove that every line intersects every smooth conic in 2 points (counting with multiplicity). If one ignores the multiplicity, which lines intersect a given conic at one point only?
(e) Fix a smooth conic $C$, a line $l$, and a point $x \in C$. Take the projection $\mathbb{P}^2 \setminus \{x\} \to l$. (A point $y \in \mathbb{P}^2 \setminus \{x\}$ is sent to the intersection point of the lines $l$, $\overline{xy}$.) Check that this gives a (continuous/differentiable/analytic) isomorphism $C \setminus \{x\} \approx \mathbb{k}^1$.