Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 1

(1) (a) Let $f \in \Bbbk[x_1, \dots, x_n]$ and suppose $f(a_1, \dots, a_n) = 0$ for any choice $\underline{a} \in \Bbbk^n$. Prove: if $\Bbbk$ is an infinite field then $0 = f \in \Bbbk[\underline{x}]$. What can happen over a finite field?

(b) Present the following subsets $X \subset \Bbbk^n$ as $X = V(I)$, i.e. find (some) defining ideals. (Compute their radicals?)
i. $X = \{(1,0), (-1,0), (0,1)\} \subset \Bbbk^2$ ii. $X = \{y = 0\} \cup \{(0,1)\} \subset \Bbbk^2$
iii. $X = \{x = 0 = y\} \cup \{y = 0 = z\} \cup \{x = 0 = z\} \subset \Bbbk^3$ iv. The image of the map $\Bbbk \xrightarrow{\phi} \Bbbk^3$, $\phi(t) = (t^3, t^4, t^5)$.
(Here the defining equations can be presented as $rank \begin{bmatrix} x & y & z \\ y & z & x^2 \end{bmatrix} < 2$. The presentation of equations via minors is not an accident, but this is beyond our course)

(c) For each ideal I of part (b) consider the ring ${}^{\Bbbk[x_1, \dots, x_n]}/_{I}$. For which cases is this ring an integral domain/ normal/local/semi-local/Artinian?

(d) For any collection of ideals $\{I_\alpha\}$ prove: $V(\cap I_\alpha) = \cup V(I_\alpha)$, $V(\sum I_\alpha) = \cap V(I_\alpha)$.

(e) Given $V(I) \subset \Bbbk^n$ and $V(J) \subset \Bbbk^m$, prove that $V(I) \times V(J) \subset \Bbbk^n \times \Bbbk^m$ is an algebraic subset. What is its defining ideal?

(f) Suppose $\Bbbk$ is a finite field, show that any subset of $\Bbbk^n$ is algebraic. Describe $I(\mathbb{A}^1) \subset \Bbbk[x]$.

(g) Which of the following subsets are algebraic?
i. $\{(cos(t), sin(t)) |\ t \in \mathbb{C}\} \subset \mathbb{C}^2$. ii. $\{r = sin(\theta)\} \subset \mathbb{R}^2$ (in polar coordinates). iii. $\{y = sin(x)\} \subset \mathbb{R}^2$.

(h) Suppose $\Bbbk$ is an infinite field and $f \in \Bbbk[x_1, \dots, x_n]$ a non-constant polynomial, $n \geq 1$. Prove: the set $\Bbbk^n \setminus V(f)$ is infinite. Suppose $n \geq 2$ and $\Bbbk$ is algebraically closed, prove: the set $V(f)$ is infinite.

(i) (Dis)prove: i. If $I, J \subset \Bbbk[\underline{x}]$ then $I \subset J$ iff $V(I) \supset V(J)$. ii. $I(X)$ is a radical ideal in $\Bbbk[\underline{x}]$.
iii. $V(I(X)) = X$ iv. $I(V(J)) = J$ v. $V(I(V(J))) = V(J)$ vi. $I(V(I(X))) = I(X)$

(2) (a) Go over all the details in the proof of Hilbert basis theorem. In particular check: for any ideal $I \subset R[x]$ and the set $I_{deg=j} = \{p \in I,\ deg(p) = j\}$, the subset $\{l.c.(I_{deg=j}), 0\} \subset R$ is an ideal. (l.c.=leading coefficients)

(b) Suppose R is Noetherian. Prove that the following rings are Noetherian:
i. ${}^{R}/_{J}$, for an ideal. ii. $R_\mathfrak{p}$, for an ideal $\mathfrak{p}$. iii. $R[[x_1, \dots, x_n]]$
(Here can use: for an ideal $J \subset R$ the completion $\hat{R}^{(J)}$ is Noetherian)

(c) Suppose R is Noetherian and $S \subset R$ is a subring. Is S necessarily Noetherian? (Give a counterexample with $R = \Bbbk[x, y]$.)

(d) Let $C^\infty(\mathcal{U})$ be the ring of all the infinitely differentiable functions on $\mathcal{U} \subset \mathbb{R}^n$. Is this ring Noetherian?

(3) (a) Find/describe the irreducible components in the following cases. Does the decomposition depend on $\Bbbk$ being algebraically closed?
i. $V(y^2 - xy - x^2y + x^3) \subset \Bbbk^2$. ii. $V(x^3 + x - x^2y - y) \subset \Bbbk^2$. iii. $V(xz, yz) \subset \Bbbk^3$
iv. $V(x^2 + y^2 - 1, x^2 - z^2 - 1) \subset \Bbbk^3$, here $\Bbbk \in \mathbb{R}, \mathbb{C}$ v. $V(xy - z^2, x^2 + y^2 + z^2) \subset \Bbbk^3$, here $\Bbbk \in \mathbb{R}, \mathbb{C}$
vi. $V\big(y^2 - x(x-1)(x-\lambda)\big) \subset \Bbbk^2$, $\lambda \in \Bbbk$, here $\Bbbk \in \mathbb{R}, \mathbb{C}$.

(b) Let $\mathfrak{S}$ be a non-empty collection of ideals in a Noetherian ring R. Prove that $\mathfrak{S}$ has a minimal element, i.e. there exists $I \in \mathfrak{S}$ which is not properly contained in any other ideal of $\mathfrak{S}$.

(c) Prove: every proper ideal in a Noetherian ring is contained in a maximal ideal.

(d) Prove: $\sqrt{I} \subset \Bbbk[\underline{x}]$ is prime iff $V(I)$ is irreducible.

(e) Prove: any radical ideal $J \subset \Bbbk[\underline{x}]$ admits the unique finite decomposition into primes, $J = \cap \mathfrak{p}_i$.

(f) For which fields is $\Bbbk^n$ irreducible?

(4) (a) Fix a polynomial map $\Bbbk^n \xrightarrow{\phi} \Bbbk^m$, and consider its graph, $\Gamma_\phi := \{(\underline{x}, \phi(\underline{x}))\} \subset \Bbbk^{n+m}$. Write down the defining ideal of Γ_ϕ and the coordinate ring $\Bbbk[\Gamma_\phi]$. Prove: $\Bbbk[\Gamma_\phi] \approx \Bbbk[x_1, \dots, x_n]$. Show the (natural) isomorphism $\Gamma_\phi \xrightarrow{\sim} \Bbbk^n$.

(b) Define the map $\Bbbk^1 \xrightarrow{\phi} \Bbbk^n$ by $\phi(t) = (t, t^2, \dots, t^n)$. Denote the image by C. Write down the defining ideal of C and the coordinate ring. Verify: $\Bbbk[C] \approx \Bbbk[t]$. Prove the isomorphism of algebraic sets $\Bbbk^1 \xrightarrow{\sim} C$.

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Homework 2

(1) (a) For $\Bbbk$ infinite prove that the curve $\{x^2+y^2=1\}\subset\Bbbk^2$ has infinitely many points. (wiki: Pythagorean triple)
   (b) Suppose $\Bbbk\subseteq\mathbb{R}$. Show that every algebraic subset of $\Bbbk^n$ can be defined by one equation, $\{f(\underline{x})=0\}\subset\Bbbk^n$.
   (c) Let $p(x,y),q(x,y)\in\Bbbk[x,y]$ be polynomials with no common factors. Prove: the set $\{p(x,y)=0=q(x,y)=0\}\subset\Bbbk^2$ is finite. (We have seen this in the class.)
   (d) Let $\Bbbk=\bar{\Bbbk}$. Prove: $V(I)\subset\Bbbk^n$ is finite iff $dim_{\Bbbk}\,\Bbbk[\underline{x}]/_I<\infty$. In this case: $\sharp(V(I))\leq dim_{\Bbbk}\,\Bbbk[\underline{x}]/_{\sqrt{I}}$. What can happen over $\mathbb{R}$? (We did not prove this in the class. If you're stuck, see [Fulton])

(2) Here we assume $\Bbbk=\bar{\Bbbk}$.
   (a) Suppose a curve $C\subset\Bbbk^2$ of degree d has a point of multiplicity d. What are the possible irreducible decompositions of C?
   (b) Fix some $m=\sum r_i$ and pairwise independent linear forms $\{l_i(x,y)\}$. Prove: for any $d>m$ there exists an irreducible curve $C\subset\Bbbk^2$, of degree d, whose tangent cone at the origin is $\{\prod l_i^{r_i}(x,y)=0\}$. (Prove: if f_m,f_d are homogeneous polynomials with no common factors then the polynomial f_m+f_d is irreducible.)

(3) (a) Let $char(\Bbbk)=0$. Show that an irreducible plane curve can have only a finite number of singular points. (This holds also in positive characteristic, then need some additional arguments.)
   (b) Let $\Bbbk=\bar{\Bbbk}$. Identify the tangent cones of the following curves at all the singular points.
      i. $V\Big((x^2+y^2-1)(x-1)(y-x-1)x\Big)$.   ii. $V\Big((x^2+y^2)^2+3x^2y-y^3\Big)$.   iii. $V\Big((x^2+y^2)^3-4x^2y^2\Big)$.
   (c) Let $\Bbbk\in\mathbb{R},\mathbb{C}$ and $pt\in C\subset\Bbbk^2$ a smooth point. Prove that the tangent line is the limit of the secants:
$$T_{(C,pt)}=\lim_{C\ni(x,y)\to pt}\overline{(x,y),pt}.$$
   (d) Let $\Bbbk=\mathbb{C}$ and $pt\in C\subset\mathbb{C}^2$ a singular point. Prove that the tangent cone, as a set, is the union of all the limits of the secants: $\{\lim\limits_{C\ni(x,y)\to pt}\overline{(x,y),pt}\}$. Does this hold also for $\Bbbk=\mathbb{R}$?
   (e) Suppose $\phi\circlearrowright\Bbbk^2$ is a change of variables, i.e. a polynomial automorphism $(x,y)\to(\tilde{x}(x,y),\tilde{y}(x,y))$. Prove that $T_{(C,pt)}\subset\Bbbk^2$ and $T_{(\phi(C),\phi(pt))}\subset\Bbbk^2$ are related by a linear transformation.

(4) (a) Let $C=\{\prod l_i(x,y)=0\}\subset\Bbbk^2$, where $l_i(x,y)$ are polynomials of degree 1, pairwise linearly independent. (Such curves are called "line arrangements".) Let $pt\in C$ a smooth point. Identify $\mathcal{O}_{(C,pt)}$.
   (b) Let $\mathfrak{p}\subset R$ be a prime ideal, check that $R_{\mathfrak{p}}$ is a local ring. Check that the natural map $R\to R_{\mathfrak{p}}$, $a\to\frac{a}{1}$ is a homomorphism of rings.
   (c) Prove: if R is Noetherian/domain/PID then so is $R_{\mathfrak{p}}$. Show that the converse does not always hold.
   (d) Let $\mathfrak{m}\subset R$ a maximal ideal. Prove that the ring $R/_{\mathfrak{m}^d}$ is local, for any $d>0$. Describe the invertible elements.
   (e) Suppose R is local and $\mathfrak{m}\subset R$ is the maximal ideal. Is $R_{\mathfrak{m}}\approx R$?
   (f) Suppose $V(I)\subset\Bbbk^n$ does not pass through the origin. What can you say about the image of I in $\Bbbk[\underline{x}]_{(\underline{x})}$? What is the geometric interpretation?
   (g) Fix an algebraic subset $X\subset\Bbbk^n$ and a polynomial automorphism $\phi\circlearrowright\Bbbk^n$. Prove: ϕ induces the isomorphisms $\Bbbk[\phi(X)]\overset{\phi^*}{\xrightarrow{\sim}}\Bbbk[X]$ and $\mathcal{O}_{(\phi(X),\phi(pt))}\overset{\phi^*}{\xrightarrow{\sim}}\mathcal{O}_{(X,pt)}$, for any $pt\in X$.
   (h) Do all the local coordinate changes arrive from the global ones?

(5) (a) Let R be a DVR and t_1,t_2 two uniformizers. Prove: $t_1=ut_2$ for some $u\in R^\times$. Prove that the valuation/order function $R\overset{ord}{\to}\mathbb{N}\cup\infty$ does not depend on the choice of uniformizer.
   (b) Check that the following rings are DVR. Give several examples of uniformizers.
      i. $\Bbbk[[x]]$.   ii. $\Bbbk\{x\}$ (for $\Bbbk\in\mathbb{R},\mathbb{C}$).   iii. $\Bbbk[x,y]_{(x,y)}/_{(y+y^3-x^3)}$.
      iv. $\Bbbk[x]_{(\infty)}:=\{\frac{p}{q}|\ p,q\in\Bbbk[x],\ q\neq0,\ deg(p)\leq deg(q)\}$. (this is called: "localization at infinity")
   (c) Let R be a DVR with the maximal ideal $\mathfrak{m}$.
      (i) Prove: $\mathfrak{m}^j/_{\mathfrak{m}^{j+1}}$ is a vector space over a field $R/_{\mathfrak{m}}$, for any $j\geq0$. Compute $dim\,\mathfrak{m}^j/_{\mathfrak{m}^{j+1}}$.
      (ii) For any $f\in R$ prove: $ord(f)=dim\,R/_{(f)}$.
      (iii) Take the quotient field $Frac(R)$. Suppose $f\in Frac(R)$. Prove: if $f\notin R$ then $\frac{1}{f}\in\mathfrak{m}\subset R$.