Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 10

(1) (a) Go over all the details of the definition of $Bl_0\mathbb{k}^n \xrightarrow{\pi} \mathbb{k}^n$. Cover $Bl_0(\mathbb{k}^n)$ by charts, each isomorphic to $\mathbb{k}^n$. Write down the transition functions. Prove that the exceptional locus of π is a hypersurface in $Bl_0\mathbb{k}^n$ and is isomorphic to $\mathbb{P}^{n-1}$.
Note: for $n = 2$, $\mathbb{k} = \mathbb{R}$, in both coordinate charts the defining equations of $Bl_0(\mathbb{R}^2)$ resemble those of the saddle point of calculus, e.g. $y = x\frac{\sigma_y}{\sigma_x}$. Use this to visualize $Bl_0(\mathbb{R}^2)$.

(b) Realize $Bl_0(\mathbb{k}^n)$ as the closure of the graph of the projection $\mathbb{k}^n \setminus \{0\} \xrightarrow{f} \mathbb{P}(\mathbb{k}^{n-1})$, $f(\underline{x}) = [\underline{x}]$. And similarly for $Bl_0\mathbb{P}^n$. Here the target $\mathbb{P}(\mathbb{k}^{n-1})$ can be thought as the infinite hyperplane in $\mathbb{P}^n$. Accordingly, $Bl_0\mathbb{P}^n$ is covered by non-intersecting lines.

(c) Let $l_1, l_2 \subset \mathbb{k}^n$ be two (distinct) lines through the origin. What are their strict transforms, $\tilde{l}_1, \tilde{l}_2 \subset Bl_0(\mathbb{k}^n)$? Compare the intersection multiplicities $i(l_1, l_2)$, $i(\tilde{l}_1, \tilde{l}_2)$.

(d) Can you imagine some version of Bezout theorem for $Bl_0\mathbb{P}^2$?

(e) Can you compute $H_*(Bl_0\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$?

(2) (a) Write down the defining equation(s) of the total and strict transforms of the curve singularities $V(x^p - y^p)$, $V(y^p - x^q)$, $p < q$, under the blowup $Bl_0(\mathbb{k}^2) \xrightarrow{\pi} \mathbb{k}^2$.

(b) Prove: the strict transform of the union of curves is the union of the strict transforms, $\widetilde{(\cup C_i, 0)} = \cup(\tilde{C}_i, 0)$.

(c) Suppose an algebraic/holomorphic map of the formal germs, $(\mathbb{k}^1, 0) \to (C, 0)$, is 1:1 an surjective. Prove that this corresponds to the embedding $\mathcal{O}_{(C,0)} \subset \mathbb{k}[[t]]$ such that $dim_{\mathbb{k}} \mathbb{k}[[t]]/\mathcal{O}_{(C,0)} < \infty$.

(d) Given a branch $(C, 0)$, with tangent $\hat{y}$, and the parametrization $(x(t), y(t))$. Prove that the parametrization of $\tilde{C}$ is $(x(t), \frac{y(t)}{x(t)})$.

(e) Go over all the detail of the resolution for branches. In the lecture we skipped the case: after several blowups one gets to $q - pj = p$. What should be done in this case?

(f) Given two smooth curve germs, $(C_1, 0), (C_2, 0) \subset (\mathbb{k}^2, 0)$, with tangency of order k, i.e. $i_0(C_1, C_2) = k$. What is the minimal number of blowups needed to separate these curves?

(3) (a) Draw the embedded resolution of the singularities $y^p = x^p$, $y^p = x^{pk}$, $y^p = x^{p+1}$.

(b) Resolve the singularity $y^5 + y^2x^2 + x^5 = 0$ and compute its Milnor number.

(4) (a) Prove that any algebraic/holomorphic map $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ is presentable as $(z_0 : z_1) \to (p(z_0, z_1) : q(z_0, z_1))$, for some homogeneous polynomials of the same degree d. What is the degree of this map?

(b) An invertible algebraic/holomorphic map $C \xrightarrow{f} C$ is called an automorphism. (Prove that f is an automorphism iff its degree is 1.) The set of all the automorphism of a curve/Riemann surface is a group, $Aut(C)$.

(c) Fix a matrix $A \in GL(2, \mathbb{k})$. Prove that the map $(z_0 : z_1) \to A(z_0 : z_1)$ defines an automorphism of $\mathbb{P}^1$. Prove that this correspondence defines a surjective homomorphism of groups $GL(2, \mathbb{k}) \twoheadrightarrow Aut(\mathbb{P}^1)$. What is the kernel?

(d) Fix a compact Riemann surface X and a meromorphic function f. Prove that f has at least one zero and at least one pole. Prove: if f has precisely one pole (or zero) then it is an isomorphism $f : X \xrightarrow{\sim} \mathbb{P}^1$.

(e) Suppose a compact Riemann surface X admits a meromorphic function f with precisely one simple pole. Prove: $f : X \xrightarrow{\sim} \mathbb{P}^1$.

(5) Let $C_1, C_2 \subset \mathbb{P}^2$ be projective curves of degree d. Suppose $C_1 \cap C_2 = d^2$ distinct points, of which $m \cdot d$ points lie on a curve of degree $m < d$ that is not a component of C_1, C_2. Prove: the other $(d - m)d$ points lie on a curve of degree $(d - m)$.