# Introduction to Algebraic Curves

201.2.4451. Summer-Fall 2019 (D.Kerner)

## Homework 11

$X$ is either a smooth projective algebraic curve (over $\Bbbk = \bar{\Bbbk}$) or a compact Riemann surface.
Notations: $L(D) := H^0(\mathcal{O}_X(D), X)$, $l(D) := h^0(\mathcal{O}_X(D))$.

(1) (a) Let $X \subset \mathbb{P}^n$ an $\Bbbk[X]$ the homogeneous coordinate ring. Let $0 \neq f \in \Bbbk[X]$. Prove: $ord_{pt}(f)$ is well defined (and finite) for any $pt \in X$. (Even though $f$ is not a function on $X$.)
  (b) Let $0 \neq f \in \Bbbk(X), \mathcal{M}_X, \Bbbk[X]$. Check that $div(f)$ is a finite sum.
  (c) We have proved in the class: if $0 \neq f \in \Bbbk(X), \mathcal{M}_X$ then $deg(div(f)) = 0$. Go over all the details of the proof.
  (d) Given a morphism $X \xrightarrow{\pi} C \subset \mathbb{P}^2$, prove: $D \equiv D' \in Div(X)$ iff $D + div(\pi^* g) = D' + div(\pi^* g')$ for some $V(g), V(g') \subset \mathbb{P}^2$ of the same degree.

(2) (a) Fix a birational model, $X \stackrel{birat}{\to} \mathbb{P}^2$, so that $\Bbbk(X) \approx \Bbbk(C)$. Then any element of $\Bbbk(C)$ is presentable in the form $\frac{p}{q}|_C$, where $p, q \in \Bbbk[x, y, z]$ are homogeneous, of the same degree. Does this imply that for any $g \in \Bbbk(X)$ the total number of zeros/poles of $g$ (counted with multiplicity) is a multiple of $deg(C)$?
  (b) Let $X = \{y^2 z = x(x-z)(x - \lambda \cdot z)\} \subset \mathbb{P}^2$, $0, 1 \neq \lambda \in \Bbbk$. Compute $div(\frac{x}{z}|_X)$, $div(\frac{y}{z}|_X)$.
  (c) (Corollary of Noether's $AF + BG$) Let $X \subset \mathbb{P}^2$ and $D_1 \equiv D_2 \in Div(X)$.
    (i) Prove: $div(q_1) + D_1 = div(q_2) + D_2$, for some $q_1, q_2 \in \Bbbk[x, y, z]$.
    (ii) Take any $p_1 \in \Bbbk[x, y, z]$ such that $D_1 \leq div(p_1)$. Apply Noether's theorem to $V(p_1 q_1), V(q_2), X \subset \mathbb{P}^2$ to get: $div(p_1 q_1) = div(p_2 q_2)$, for some $p_2 \in \Bbbk[x, y, z]$.
    (iii) Prove: if $D_1 \leq div(p_1)$ for some $p_1 \in \Bbbk[x, y, z]$, then $div(p_1) - D_1 = div(p_2) - D_2$, for some $p_2 \in \Bbbk[x, y, z]$.
  (d) (i) Let $C \subset \mathbb{P}^2$ a smooth cubic and $pt_1, pt_2 \in C$. Prove: $C \not\approx \mathbb{P}^1$.
    (You can either use question 3(d) or verify: $pt_1 \equiv pt_2$ iff $pt_1 = pt_2$.)
    (ii) Demonstrate a basis of $L(D)$ for $deg(D) = 1, 2, 3$. ('Explicit', as in question 4(a).) What happens when the points of $D$ are non-distinct? For $deg(D) = 3$ what happens when the points of $D$ lie on a line?
    (iii) Let $pt \in C$ a flex. Demonstrate a basis of $L(3n \cdot pt)$, for any $n \geq 1$. (Conclude again that $g(C) = 1$.)

(3) (a) Let $D = \sum_{finite} n_i pt_i \in Div(\mathbb{P}^1)$. Write down an explicit basis of $L(D)$.
  (b) Let $C \subset \mathbb{P}^2$ be a singular, irreducible cubic and $p, q \in C$ some smooth points. Prove: $p \equiv q$.
  (c) Suppose $p \equiv q$ for some distinct points $p, q \in X$. Prove: $X \xrightarrow{\sim} \mathbb{P}^1$.
  (d) (i) Let $0 \leq D = \sum n_{pt} pt \in Div(X)$ and $S_D = \{pt \in X | \ n_{pt} > 0\}$. Prove: $L(D) \subseteq \mathcal{O}_X(X \setminus S_D)$.
    (ii) Fix an open subset $\varnothing \neq \mathcal{U} \subsetneq X$. Prove: $dim_{\Bbbk}(\mathcal{O}_X(\mathcal{U})) = \infty$.
    (iii) Prove: $l(D) > 0$ iff $D \equiv D'$ where $D'$ is effective (i.e. $D' = \sum n_{pt} pt$ with $n_{pt} \geq 0$).
    (iv) Prove: for any $D_1, D_2 \in Div(X)$ with $D_1 < D_2$ holds: $l(D_2) \leq l(D_1) + deg(D_2) - deg(D_1)$.
    (v) For $g(X) = 1$ prove: $l(D) = deg(D)$, for any $D$ with $deg(D) \geq 1$.
    (vi) Let $l(D) > 0$ and $0 \neq f \in L(D)$. Show: $f \in L(D - pt)$ for at most a finite number of points on $X$.
    (vii) Thus $l(D - pt) = l(D) - 1$ for "almost all" points of $X$ (i.e. except for a finite subset).
    (viii) Prove: if $l(pt_1 + pt_2) = 2$ for <u>any</u> $pt_1, pt_2 \in X$ then $g(X) = 1$.
    (Achtung: this does not hold if 'any' is replaced by 'some'.)

(4) (a) Let $Div_0(X) < Div(X)$ be the subgroup of all the divisors of total degree zero. Prove: $\Bbbk(X) \setminus \{0\} \stackrel{div}{\to} Div_0(X)$ is a homomorphism of abelian groups. Describe its kernel.
  (b) Check that the relation $D \equiv D'$ on divisors is indeed an equivalence relation. Check that this relation is compatible with the group structure of $Div_0(X)$, $Div(X)$ and the set $\{D | \ D \equiv 0\}$ is a subgroup of $Div_0(X)$.
  (c) Check that ${}^{Div_0(X)}/_{\equiv}$, ${}^{Div(X)}/_{\equiv}$ are abelian groups. Prove: ${}^{Div(\mathbb{P}^1)}/_{\equiv} \approx \mathbb{Z}$. Prove that ${}^{Div_0(X_{g=1})}/_{\equiv}$ is not finitely generated.
  (d) Fix a smooth cubic $C \subset \mathbb{P}^2$ and a point $p \in C$, this defines the group structure. Prove that the map $x \to x - p$ is an isomorphism of groups, $(C, +, p) \approx {}^{Div_0(X_{g=1})}/_{\equiv}$.

(5) Finally we can prove: any irreducible complex algebraic curve is connected in the classical topology.
  (a) Prove: it is enough to consider only smooth projective (complex, irreducible) curves. (By desingularization.)
  (b) Let $X$ as above. Prove: $X$ has no isolated points. (Map $X$ birationally into $\mathbb{P}^2$)
  (c) Suppose $X$ as above has two connected components, $X_1, X_2$. Both are compact (in the classical topology). Fix a point $pt \in X_1$ and take some $f \in \mathcal{M}_X(X)$ that has a pole only at $pt$. (Why does such a function exist?) Then $f|_{X_2}$ is a holomorphic function on a compact set, thus is a constant. Thus we can assume $f|_{X_2} = 0$. But then $X_2 \subset X$ is an algebraic subset.
  (d) Prove: any complex projective curve, $C \subset \mathbb{P}^2$, (not necessarily irreducible) is connected.
  Does this hold also for an affine curve $C \subset \mathbb{C}^2$?