Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 2

(1) (a) For $\Bbbk$ infinite prove that the curve $\{x^2+y^2=1\}\subset\Bbbk^2$ has infinitely many points. (wiki: Pythagorean triple)
   (b) Suppose $\Bbbk\subseteq\mathbb{R}$. Show that every algebraic subset of $\Bbbk^n$ can be defined by one equation, $\{f(\underline{x})=0\}\subset\Bbbk^n$.
   (c) Let $p(x,y),q(x,y)\in\Bbbk[x,y]$ be polynomials with no common factors. Prove: the set $\{p(x,y)=0=q(x,y)=0\}\subset\Bbbk^2$ is finite. (We have seen this in the class.)
   (d) Let $\Bbbk=\bar{\Bbbk}$. Prove: $V(I)\subset\Bbbk^n$ is finite iff $dim_{\Bbbk}\,{}^{\Bbbk[\underline{x}]}/_{I}<\infty$. In this case: $\sharp(V(I))\le dim_{\Bbbk}\,{}^{\Bbbk[\underline{x}]}/_{\sqrt{I}}$. What can happen over $\mathbb{R}$? (We did not prove this in the class. If you're stuck, see [Fulton])

(2) Here we assume $\Bbbk=\bar{\Bbbk}$.
   (a) Suppose a curve $C\subset\Bbbk^2$ of degree d has a point of multiplicity d. What are the possible irreducible decompositions of C?
   (b) Fix some $m=\sum r_i$ and pairwise independent linear forms $\{l_i(x,y)\}$. Prove: for any $d>m$ there exists an irreducible curve $C\subset\Bbbk^2$, of degree d, whose tangent cone at the origin is $\{\prod l_i^{r_i}(x,y)=0\}$. (Prove: if f_m,f_d are homogeneous polynomials with no common factors then the polynomial f_m+f_d is irreducible.)

(3) (a) Let $char(\Bbbk)=0$. Show that an irreducible plane curve can have only a finite number of singular points. (This holds also in positive characteristic, then need some additional arguments.)
   (b) Let $\Bbbk=\bar{\Bbbk}$. Identify the tangent cones of the following curves at all the singular points.
      i. $V\Big((x^2+y^2-1)(x-1)(y-x-1)x\Big)$. ii. $V\Big((x^2+y^2)^2+3x^2y-y^3\Big)$. iii. $V\Big((x^2+y^2)^3-4x^2y^2\Big)$.
   (c) Let $\Bbbk\in\mathbb{R},\mathbb{C}$ and $pt\in C\subset\Bbbk^2$ a smooth point. Prove that the tangent line is the limit of the secants:

$$T_{(C,pt)}=\lim_{C\ni(x,y)\to pt}\overline{(x,y),pt}.$$

   (d) Let $\Bbbk=\mathbb{C}$ and $pt\in C\subset\mathbb{C}^2$ a singular point. Prove that the tangent cone, as a set, is the union of all the limits of the secants: $\{\lim\limits_{C\ni(x,y)\to pt}\overline{(x,y),pt}\}$. Does this hold also for $\Bbbk=\mathbb{R}$?
   (e) Suppose $\phi\circlearrowright\Bbbk^2$ is a change of variables, i.e. a polynomial automorphism $(x,y)\to(\tilde{x}(x,y),\tilde{y}(x,y))$. Prove that $T_{(C,pt)}\subset\Bbbk^2$ and $T_{(\phi(C),\phi(pt))}\subset\Bbbk^2$ are related by a linear transformation.

(4) (a) Let $C=\{\prod l_i(x,y)=0\}\subset\Bbbk^2$, where $l_i(x,y)$ are polynomials of degree 1, pairwise linearly independent. (Such curves are called "line arrangements".) Let $pt\in C$ a smooth point. Identify $\mathcal{O}_{(C,pt)}$.
   (b) Let $\mathfrak{p}\subset R$ be a prime ideal, check that $R_{\mathfrak{p}}$ is a local ring. Check that the natural map $R\to R_{\mathfrak{p}}$, $a\to\frac{a}{1}$ is a homomorphism of rings.
   (c) Prove: if R is Noetherian/domain/PID then so is $R_{\mathfrak{p}}$. Show that the converse does not always hold.
   (d) Let $\mathfrak{m}\subset R$ a maximal ideal. Prove that the ring ${}^{R}/_{\mathfrak{m}^d}$ is local, for any $d>0$. Describe the invertible elements.
   (e) Suppose R is local and $\mathfrak{m}\subset R$ is the maximal ideal. Is $R_{\mathfrak{m}}\approx R$?
   (f) Suppose $V(I)\subset\Bbbk^n$ does not pass through the origin. What can you say about the image of I in $\Bbbk[\underline{x}]_{(\underline{x})}$? What is the geometric interpretation?
   (g) Fix an algebraic subset $X\subset\Bbbk^n$ and a polynomial automorphism $\phi\circlearrowright\Bbbk^n$. Prove: ϕ induces the isomorphisms $\Bbbk[\phi(X)]\overset{\phi^*}{\xrightarrow{\sim}}\Bbbk[X]$ and $\mathcal{O}_{(\phi(X),\phi(pt))}\overset{\phi^*}{\xrightarrow{\sim}}\mathcal{O}_{(X,pt)}$, for any $pt\in X$.
   (h) Do all the local coordinate changes arrive from the global ones?

(5) (a) Let R be a DVR and t_1,t_2 two uniformizers. Prove: $t_1=ut_2$ for some $u\in R^\times$. Prove that the valuation/order function $R\overset{ord}{\to}\mathbb{N}\cup\infty$ does not depend on the choice of uniformizer.
   (b) Check that the following rings are DVR. Give several examples of uniformizers.
      i. $\Bbbk[[x]]$. ii. $\Bbbk\{x\}$ (for $\Bbbk\in\mathbb{R},\mathbb{C}$). iii. ${}^{\Bbbk[x,y]_{(x,y)}}/_{(y+y^3-x^3)}$.
      iv. $\Bbbk[x]_{(\infty)}:=\{\frac{p}{q}|\ p,q\in\Bbbk[x],\ q\neq0,\ deg(p)\le deg(q)\}$. (this is called: "localization at infinity")
   (c) Let R be a DVR with the maximal ideal $\mathfrak{m}$.
      (i) Prove: ${}^{\mathfrak{m}^j}/_{\mathfrak{m}^{j+1}}$ is a vector space over a field ${}^{R}/_{\mathfrak{m}}$, for any $j\ge0$. Compute $dim\,{}^{\mathfrak{m}^j}/_{\mathfrak{m}^{j+1}}$.
      (ii) For any $f\in R$ prove: $ord(f)=dim\,{}^{R}/_{(f)}$.
      (iii) Take the quotient field $Frac(R)$. Suppose $f\in Frac(R)$. Prove: if $f\notin R$ then $\frac{1}{f}\in\mathfrak{m}\subset R$.