Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 3

(1) (a) In the lecture we have stated the equivalence of criteria for R a domain, not a field, to be a DVR. Prove this:
  (i) R is Noetherian, local and the maximal ideal is principal.
  (ii) Exists an irreducible element $t \in R$ such that $R = \{ut^d \mid u \in R^\times,\ d \in \mathbb{N}\} \cup \{0\}$.
 (b) Suppose a DVR contains a field $\Bbbk$ such that the composition $\Bbbk \to R \to R/\mathfrak{m}$ is an isomorphism. Prove:
  (i) For any $z \in R$ exists a unique $\lambda \in \Bbbk$ such that $z - \lambda \in \mathfrak{m}$.
  (ii) Fix a uniformizing parameter $t \in R$. Then for any $z \in R$ and $n \in \mathbb{N}$ exists a unique presentation: $z = \sum_{i=0}^{n} \lambda_i t^i + z_n t^{n+1}$, with $\{\lambda_i \in \Bbbk\}$ and $z_n \in R$.
  (iii) This presentation defines an embedding (injective homomorphism) $R \hookrightarrow \Bbbk[[x]]$.
 (c) Go over all the details of the proof: the curve $C \subset \Bbbk^2$ is smooth at pt iff the ring $\mathcal{O}_{(C,pt)}$ is DVR.
 Why the proof does not imply: any DVR is isomorphic to $\mathcal{O}_{(\Bbbk^1,0)}$?
 (d) Prove: $g(x,y) \in \Bbbk[x,y]$ is sent to a uniformizer in $\mathcal{O}_{(C,pt)}$ iff the curve $V(g)$ is smooth at pt and non-tangent to C.

(2) Recall the uniqueness theorem for holomorphic functions: if $f, g \in \mathcal{O}(\mathcal{U})$ coincide on a convergent (in $\mathcal{U}$) sequence of points then they coincide on the corresponding connected component of $\mathcal{U}$. (And no control over the other connected components.) Formulate the uniqueness for localizations.

(3) (a) Let $0 \to V_1 \to \cdots \to V_n \to 0$ be an exact sequence of finite-dimensional $\Bbbk$-vector spaces. Show: $\sum_i (-1)^j dim_{\Bbbk}(V_j) = 0$.
 (b) For any triple of modules $M \subset N \subset L$ prove: the sequence $0 \to N/M \to L/M \to L/N \to 0$ is exact.
 (We will often use this for $\mathfrak{m}^{j+n} \subset \mathfrak{m}^j \subset R$.)

(4) In the proof of "$mult_{pt}(f) = \mathfrak{m}^j_{pt}/\mathfrak{m}^{j+1}_{pt}$ for $j \gg 1$", we have used the exact sequence $0 \to \mathfrak{m}^j/\mathfrak{m}^{j+1} \to \mathcal{O}_{(C,pt)}/\mathfrak{m}^{j+1} \to \mathcal{O}_{(C,pt)}/\mathfrak{m}^j \to 0$. For an arbitrary algebraic subset $X \subset \Bbbk^n$ the Hilbert-Samuel function of the local ring at $pt \in X$ is defined as $\chi(j) := dim_{\Bbbk}\, \mathcal{O}_{(X,pt)}/\mathfrak{m}^j_{pt}$.
 (a) Prove that the function $\chi(j) := dim_{\Bbbk} \mathcal{O}_{(\Bbbk^n,pt)}/\mathfrak{m}^j_{pt}$ is a polynomial in j, of degree n, whose leading coefficient is $\frac{1}{n!}$.
 (b) Let $X = \{f(\underline{x}) = 0\} \subset \Bbbk^n$ be a hypersurface. Prove: for $j \gg 1$ the function $\chi(j) := dim_{\Bbbk} \mathcal{O}_{(X,pt)}/\mathfrak{m}^j_{pt}$ is a polynomial in j, of degree $(n-1)$, whose leading coefficient is $\frac{ord_{pt}(f)}{(n-1)!}$.

(5) Here we assume $\Bbbk = \bar{\Bbbk}$.
 (a) We had stated that the definition "$i_{pt}(f_1, f_2) = dim\, \Bbbk[x,y]_{\mathfrak{m}_{pt}}/(f_1, f_2)$" satisfies all the properties of the wish list for the intersection multiplicity. Verify this.
 (b) Compute $i_0(f_1, f_2)$ for $f_1(x,y) = (x^2+y^2)^2 + 3x^2y - y^3$, $f_2(x,y) = (x^2+y^2)^3 - 4x^2y^2$. (If you are stuck, wait for the next lecture.)
 (c) Suppose $pt \in C \subset \Bbbk^2$ is a smooth point and fix a uniformizer $t \in \mathcal{O}_{(C,pt)}$. Let $D = \{f(x,y) = 0\}$. Prove: $i_{pt}(C,D) = ord_t(f(x(t), y(t)))$.
 (d) (Can postpone this till the next lecture)
 Go over all the details of the proof of "$i_{pt}(C,D) \geq mult_{pt}(C) \cdot mult_{pt}(D)$ with equality iff ...".
 In particular, prove: If $\mathfrak{m}^{m_1+m_2-1} \subseteq (f,g) + \mathfrak{m}^{m_1+m_2} \subseteq \Bbbk[x,y]_{\mathfrak{m}}$ then $\mathfrak{m}^{m_1+m_2-1} \subseteq (f,g) \subseteq \Bbbk[x,y]_{\mathfrak{m}}$.
 This is immediate if one uses the Nakayama lemma. For a proof without Nakayama see Fulton, page 39.
 (e) Let $pt \in \{f(x,y) = 0\}$ be a smooth point. Prove: $i_{pt}(f, g+h) \geq min(i_{pt}(f,g), i_{pt}(f,h))$.
 Does this hold also for non-smooth points?
 (f) Fix a curve $C = \{f(x,y) = 0\}$ and a parameterized line $L = \{(x(t), y(t)) = (a_x t + b_x, a_y t + b_y)\}$. Check that the roots of the equation $f(x(t), y(t)) = 0$ correspond to the intersection points, $C \cap L$, with their multiplicities. Prove: $\sum i_{pt}(L, C) \leq deg(f)$. When is the equality realized?
 (g) Give an example of two (smooth) conics both passing through $(0,0) \in \Bbbk^2$, whose local intersection multiplicity at $(0,0)$ equals 4.
 (h) Suppose C is smooth at pt and $L = T_{(C,pt)}$. Check: $i_{pt}(C,L) \geq 2$. This point is called an inflection point (or "a flex") if $i_{pt}(C,L) > 2$. What are the flexes of the curve $\{y = x^n\} \subset \Bbbk^2$?
  (i) Does this definition depend on the choice of local coordinates?
  (ii) Prove: a smooth conic in $\Bbbk^2$ has no flexes. (Hint: no heavy computations are needed here)
  (iii) Prove: $pt \in \{f(x,y) = 0\}$ is an inflection point iff the Hessian matrix satisfies:

$$(-\partial_y f, \partial_x f)|_{pt} \cdot \partial^2 f|_{pt} \cdot \begin{pmatrix} -\partial_y f \\ \partial_x f \end{pmatrix}|_{pt} = 0.$$