Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 4

(1) (a) Let $\{\mathcal{U}_i\}$ be the standard affine charts on $\mathbb{P}^n$. Describe the sets $X_i := \mathbb{P}^n \setminus \{\cup_{j\neq i}\mathcal{U}_j\}$, $X_{12} := \mathbb{P}^n \setminus \{\cup_{j\neq 1,2}\mathcal{U}_j\}$.
(b) Fix an affine plane of dimension r with the parametrization $L = \{\vec{v}_0 + \sum t_i\vec{v}_i | \ (t_1,\dots,t_r) \in \Bbbk^r\} \subset \Bbbk^n$.
(c) Prove: through any two points of $\mathbb{P}^2$ passes unique (projective) line.
(d) Prove: given any $\binom{d+2}{2} - 1$ points in $\mathbb{P}^2$ there is a curve of degree d passing through them. Moreover, if the points are "mutually generic" then such a curve is unique. (Here the genericity condition is formulated as the non-degeneracy of some matrix, what is this matrix?)
(e) Let $L_1, L_2 \subset \mathbb{P}^n$ be planes of dimensions d_1, d_2. Prove: $dim(L_1 \cap L_2) \geq d_1 + d_2 - n$. Does this hold also in $\Bbbk^n$? (Which condition should be added?)
(f) Let X be the one-point compactification of $\mathbb{C}^2$. Let $l_1, l_2 \subset \mathbb{C}^2$ be two non-parallel lines. In how many points do their closures intersect?
(g) Let $L_i = \{l_i(\underline{x}) = 0\}$ be two (distinct) hyperplanes in $\Bbbk^n$. (Here $deg(l_i) = 1$) Consider the spanned family $\{l_1(\underline{x}) + l_2(\underline{x}) = 0\} \subset \Bbbk^n_{\underline{x}} \times \Bbbk^1_t$.
(i) Describe the members of this family. (We did this in the class for $L_1, L_2 \subset \Bbbk^2$.) What is the base locus?
(ii) This family contains L_1, but not L_2. How to extend this family so that the value $t = \infty$ will be allowed?
(iii) Extend all this to one-dimensional families of hyperplanes in $\mathbb{P}^n$.

(2) (a) Prove: $f(\underline{x}) \in \Bbbk[x_1,\dots,x_n]$ is irreducible iff $x_0^{deg(f)} f(\frac{x_1}{x_0},..,\frac{x_n}{x_0}) \in \Bbbk[x_0,\dots,x_n]$ is irreducible.
(b) Prove: $I \subset \Bbbk[x_0,\dots,x_n]$ has a system of homogeneous generators iff for any $f \in I$ all the homogeneous components of f belong to I.
(c) Prove: an algebraic set $X \subset \mathbb{P}^n$ is irreducible iff the ideal $I(X) \subset \Bbbk[x_0,\dots,x_n]$ is prime.
(d) Let $I \subset \Bbbk[\underline{x}]$ be a homogeneous prime ideal. Prove: if $f \cdot g \in I$ then either all homogeneous components of f belong to I or the same for g. (Thus I is "homogeneously prime".)
(e) Prove: if I is a homogeneous ideal then $\sqrt{I}$ is homogeneous.
(f) Suppose $\Bbbk$ is infinite. Prove: if $X \subset \mathbb{P}^n$ is an algebraic subset and $f \in \Bbbk[x_0,\dots,x_n]$ vanishes identically on X, then all the homogeneous components of f vanish on X.
(g) Prove: any intersection/finite union of algebraic subsets of $\mathbb{P}^n$ is algebraic.
(h) Prove: $X \subset \mathbb{P}^n$ is a projective algebraic set/hypersurface iff for any affine chart $\mathcal{U}_i$: $X \cap \mathcal{U}_i$ is an affine algebraic set/hypersurface.
(i) Let $I = (x_1 + x_3^2, x_2 + x_3^2)$ and $X = V(I) \subset \Bbbk^3$. Describe $\overline{X} \subset \mathbb{P}^3$ and $I_{\overline{X}}$. How $\overline{X}$ "looks like" near L_∞? (Warning: the defining ideal of $\overline{X} \subset \mathbb{P}^3$ is not generated by just two elements.)

(3) (a) The group $\mathbb{P}GL(n+1,\Bbbk) = {}^{GL(n+1,\Bbbk)}/_{\Bbbk^\times}$ acts on $\mathbb{P}^n$, these are called: projective transformations. Prove: $\mathbb{P}GL(n+1,\Bbbk)$ sends algebraic subsets to algebraic, hypersurfaces to hypersurfaces, planes to planes, smooth points of curves to smooth points.
(b) Fix some generating linear form, $I_X = (l_1(\underline{x}),\dots,l_r(\underline{x}))$, and take the matrix of their coefficients, $A \in Mat_{r\times(n+1)}(\Bbbk)$. The codimension of $X \subseteq \mathbb{P}^n$ is the row-rank of this matrix, while the dimension is $n - codim(X)$. Check that these do not depend on the choice of coordinates in $\mathbb{P}^n$, generators of I_X. Check that the (co)dimension of a linear subset coincides with the (co)dimension of (any of) its affine part.
Convert this into the parametrization of $\bar{L} \subset \mathbb{P}^n$.
(c) Prove: any three points $pt_1, pt_2, pt_3 \in \mathbb{P}^2$ not lying on one line can be brought by $\mathbb{P}GL(3)$ to $[1:0:0]$, $[0:1:0]$, $[0:0:1]$. Formulate and prove the higher dimensional analogue of this statement.
(d) Prove: any three lines in $\mathbb{P}^2$ that do not all pass through one point can be brought to the lines $\{x_i = 0\}_{i=0,1,2}$. Formulate and prove the analogous statement for hyperplanes in $\mathbb{P}^n$.

(4) (a) Prove: any homogeneous ideal in $\Bbbk[\underline{x}]$ is homogeneously finitely generated.
(b) For a projective algebraic set $X \subset \mathbb{P}^n$ take the coordinate ring $\Bbbk[X]$. Prove: every element of $\Bbbk[X]$ can be written (uniquely) as $\sum \bar{f}_i$, where $\bar{f}_i \in \Bbbk[X]$ is the image of some homogeneous polynomial $f_i \in \Bbbk[x_0,\dots,x_n]$
Prove: the images of degree-d-homogeneous polynomials form a finite dimensional vector subspace $\Bbbk[X]_d \subset \Bbbk[X]$.
(c) Fix a hypersurface $X = \{f(\underline{x}) = 0\} \subset \mathbb{P}^n$. Check the exactness of the sequence $0 \to \Bbbk[\underline{x}] \xrightarrow{\times f} \Bbbk[\underline{x}] \to \Bbbk[X] \to 0$. Compute $dim_{\Bbbk}\Bbbk[X]_d$.
(d) Take some algebraic subsets $X \subseteq Y \subseteq \mathbb{P}^n$, with X an irreducible hypersurface.
Prove: either $X = Y$ or $Y = \mathbb{P}^n$ or Y is reducible.

(5) (a) Prove: any non-empty open subset of an irreducible algebraic subset (in $\mathbb{k}^n$, $\mathbb{P}^n$) is dense.
(b) Prove: $X \subset \mathbb{P}^n$ is closed iff $X \cap \mathcal{U}_i$ is closed for any chart $\mathcal{U}_i = \{x_i \neq 0\}$.
(c) Find the Zariski closure of the following subsets: $\mathbb{Z} \subset \mathbb{Q}$, $\{y = sin(x)\} \subset \mathbb{C}^2$.
(d) Prove: the projective closure of an affine hypersurface is a hypersurface, i.e. it is defined by one equation.

(6) (a) Prove: any continuous bijection of irreducible plane curves is a homeomorphism.
(Later we will define the notion of isomorphism, and show that a homeomorphism is not necessarily an isomorphism.)
(b) Prove: any collection of closed subsets of a (affine/projective) algebraic set has a minimal member.
(c) Prove: $\mathbb{P}^n_{\mathbb{k}}$ is compact in Zariski topology. For $\mathbb{k} \subseteq \mathbb{C}$ prove: $\mathbb{P}^n_{\mathbb{k}}$ is compact in the classical topology.
(d) Let $V \subseteq \mathbb{P}^n$ be an algebraic subset and $X \subseteq V$ be open. Prove: any open cover of X has a finite subcover.
(e) Let $X \subset \mathbb{C}^n$ be an algebraic subset, $dim(X) > 0$. Prove: X is not bounded. (In particular, not compact in the classical topology.) Therefore any (complex) projective algebraic set of positive dimension cannot be realized as an algebraic subset of $\mathbb{C}^n$, for any n. (e.g. $\mathbb{P}^1 \not\subset \mathbb{C}^n$, for any n.)