Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 5

(1) (a) In the lecture we have obtained the local ring $\mathcal{O}_{(X,pt)}$ of $X \subset \mathbb{P}^n$ by first restricting to some affine chart $\mathcal{U}_i$ and then localizating. Prove that $\mathcal{O}_{(X,pt)}$ does not depend on the choice of the affine chart.
(b) In the lecture we have defined the function field $\Bbbk(X)$. Check that this is indeed a field. For $X = \mathbb{P}^n$ compute its transcendence degree over $\Bbbk$.
(c) Let $X \subset \mathbb{P}^n$ be defined by a prime ideal $I \subset \Bbbk[\underline{x}]$ and let $f \in \Bbbk(X)$. Prove: the sets of zeros/poles of f are algebraic subsets of $\mathbb{P}^n$.

(2) (a) Suppose $\Bbbk = \bar{\Bbbk}$. Show that any smooth conic is $\mathbb{P}GL(2)$-equivalent to $V(x^2 + y^2 + z^2)$.
(b) Show that pt is a singular point of $\{f(x, y, z) = 0\} \subset \mathbb{P}^2$ iff $\partial_x f|_{pt} = \partial_y f|_{pt} = \partial_z f|_{pt} = 0$. (Assume $char(\Bbbk) \nmid deg(f)$.) If $[x_0 : y_0 : z_0]$ is a smooth point, check that the tangent line is defined by $(x - x_0)\partial_x f + (y - y_0)\partial_y f + (z - z_0)\partial_z f = 0$.
(c) For any $pt \in C = \{f(x, y, z) = 0\} \subset \mathbb{P}^2$ prove: $mult_{pt}(C) \leq mult_{pt}\{\partial_x f(x, y, z) = 0\} + 1$.
(d) Find all the intersection points and the local intersection multiplicities of $V(y^2 z - x(x - 2z)(x + z))$, $V(y^2 + x^2 - 2xz)$ in $\mathbb{P}^2$.
(e) (Here $char(\Bbbk) = 0$.) Suppose the curve $C = V(f) \subset \mathbb{P}^2$ is irreducible. Prove: $\partial_x f|_C \cdot \partial_y f|_C \cdot \partial_z f|_C \not\equiv 0$. In particular C has a finite number of singular points.
(f) (Here $\Bbbk = \bar{\Bbbk}$) Prove: a plane cubic with more than one singular point is reducible. Classify, up to $\mathbb{P}GL(3)$-equivalence, all the reducible plane cubics.

(3) Fix a local ring $(R, \mathfrak{m})$. The order of an element $f \in R$ is $ord(f) := sup\{j |\ f \in \mathfrak{m}^j\} \leq \infty$. (Dis)prove the following properties. Even if a property fails in general, give several examples of local rings where this property holds.
i. $ord(f) = \infty$ iff $f = 0$. ii. $ord(f \pm g) \geq min(ord(f), ord(g))$. iii. $ord(f \cdot g) = ord(f) + ord(g)$
iv. $ord(\frac{f}{g}) = ord(f) - ord(g)$.

(4) (a) Suppose $\Bbbk$ is infinite. Prove: for any finite subset $S \subset \mathbb{P}^2$ there exists a curve $C \subset \mathbb{P}^2$ such that $C \cap S = \varnothing$.
(b) ($\Bbbk = \bar{\Bbbk}$) Fix an irreducible curve $C \subset \mathbb{P}^2$, some smooth points on it, $pt_1, \dots, pt_r$, and some integers $\{m_i\}_i$. Construct a rational function $f \in \Bbbk(C)$ with $ord_{pt_i}(f) = m_i$.
(c) ($\Bbbk = \bar{\Bbbk}$) Prove that every non-singular curve $V(f) \subset \mathbb{P}^2$ is irreducible. (What about the affine plane curves?)

(5) (a) Let $p(x) = \prod (x - x_i)^{n_i}$. Prove: $\Bbbk[x]/_{p(x)} \xrightarrow{\sim} \prod \Bbbk[x]/_{(x - x_i)^{n_i}}$. (This is a particular case of the general statement proven in the class, but in the one-variable case the proof is straightforward.)
(b) ($\Bbbk = \bar{\Bbbk}$) Fix a finite collection of points $\{pt_i\}$ in $\mathbb{P}^n$. For any $d \in \mathbb{N}$ prove: $(\cap \mathfrak{m}_{pt_i})^d = \cap \mathfrak{m}^d_{pt_i}$. (see Fulton, pg.26)

(6) ($char(\Bbbk) = 0$, $\Bbbk = \bar{\Bbbk}$) The following continues question 5 of hwk 3. Fix a smooth curve $V(f) \subset \mathbb{P}^2$ and consider the Hessian matrix, $\partial^2 f \in Mat_{3\times 3}(\Bbbk[x, y, z])$.
(a) Show: if $deg(f) > 2$ then $det(\partial^2 f)$ is non-constant. (Hint: you can use the Euler formula $\sum x_i \partial_i f = d \cdot f$.)
(b) Prove: the points of $V(f) \cap V(det(\partial^2 f))$ are exactly the flexes of $V(f)$. Moreover: $pt \in V(f)$ is an ordinary flex iff $i_{pt}(V(f), V(det(\partial^2 f))) = 1$.
(Hint: apply $\mathbb{P}GL(3)$ to set $pt = (0 : 0 : 1)$ with the tangent line $(y = 0)$. (How does this affect $\partial^2 f$?) Then we can assume $f(x, y, z) = z^{d-1} y + a z^{d-2} x^2 + y^2(\dots) + xy(\dots) + x^3(\dots)$. Compute $\partial^2 f|_{(0:0:1)}$.)
(c) Conclude: every smooth curve of $deg > 2$ has a flex.
Prove: if all the flexes of a smooth curve are ordinary then their total number is $d \cdot 3(d - 2)$.

(7) Assume $\Bbbk = \bar{\Bbbk}$, $char(\Bbbk) = 0$. We classify the irreducible plane cubics.
(a) Show that a smooth plane cubic has only ordinary flexes. Conclude: a smooth cubic has precisely 9 flexes.
(b) Put one of the flexes to $(0 : 1 : 0)$ with the tangent line $(z = 0)$. Then the defining equation consists of monomials $y^2 z$, yxz, x^3, $x^2 z$, xz^2, z^3. Apply now a $\mathbb{P}GL(3, \Bbbk)$ transformations to get rid of yzx, z^3. Then arrive at the Weierstraß normal form of smooth cubic: $y^2 z = x(x - z)(x - \lambda \cdot z)$, with $\lambda \neq 0, 1$.
(c) Prove: any irreducible singular cubic is $\mathbb{P}GL(3)$-equivalent to one of: $y^2 = x^3$, $y^2 = x^2 + x^3$.

(8) Fix some $h, f, g \in \mathfrak{m} = (x, y) \subset \Bbbk[x, y]$. Prove: any of the following conditions ensures $(h)_{\mathfrak{m}} \in (f, g)_{\mathfrak{m}} \subset \Bbbk[x, y]_{\mathfrak{m}}$.
(a) $V(f), V(g)$ are transverse at 0 (i.e. both are smooth and non-tangent) and $h|_0 = 0$.
(b) $V(f)$ is smooth at 0 and $i_0(h, f) \geq i_0(g, f)$.
(c) $T_{(V(f),0)} \cap T_{(V(g),0)} = \{0\}$ and $ord_0(h) \geq ord_0(f) + ord_0(g) - 1$.