Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 6

(1) (a) Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth (projective, complex) curve. Prove: C is a real orientable compact manifold.
  (b) Let $C = V(y^2 z - x^3) \subset \mathbb{P}^2$. Prove: $\Bbbk(C) \cong \Bbbk(\mathbb{P}^1)$. For $\Bbbk = \mathbb{C}$ give an explicit homeomorphism $C \approx S^2$.
  (c) Suppose a curve $C \subset \mathbb{P}^2$ is irreducible and $f \in \Bbbk(C)$ is regular at each point of C. Prove: f is constant.
  (d) We have proved: if $X = \coprod X_i$, where $\{X_i\}$ are one-point sets, then $\Bbbk[X] \approx \prod \mathcal{O}_{(X,pt_i)}$. In particular, $\Bbbk[X]$ does not depend on the location of these points. Does this contradict the known fact: "two quadruples $\{x_1, x_2, x_3, x_4\} \subset \mathbb{C}^1$ are related by an analytic automorphism of $\mathbb{C}$ iff the corresponding cross-ratios coincide"?

(2) (a) Suppose two cubics intersect at nine distinct points, $\{pt_i\}$. Suppose $pt_1, \dots, pt_6$ lie on one conic. Prove: pt_7, pt_8, pt_9 lie on a line.
  (b) (Pascal theorem) Fix six points on a smooth conic, $p_1, p_2, p_3, q_1, q_2, q_3$. Consider the intersection points of lines, $z_{ij} := \{\overline{p_i q_j} \cap \overline{p_j q_i}\}_{i \neq j}$. Prove: z_{12}, z_{13}, z_{23} lie on one line.
  (c) (Pappus theorem) Fix two lines, $L, \tilde{L} \subset \mathbb{P}^2$ and the points $p_1, p_2, p_3 \in L$ and $q_1, q_2, q_3 \in \tilde{L}$, none of which is the intersection point $L \cap \tilde{L}$. Prove: the points $\{\overline{p_i q_j} \cap \overline{p_j q_i}\}_{i \neq j}$ lie on one line.
  (d) Suppose two cubics intersect at the distinct points $pt_1, \dots, pt_9$ and a third cubic passes through $pt_1, \dots, pt_8$. Prove: it also passes through pt_9.

(3) We have defined the group law on a smooth cubic, $C \subset \mathbb{P}^2$. Check that $(C, +, 0)$ is indeed a commutative group. (For associativity see pg. 63-64 of Fulton.)
  (a) Fix a line L such that $L \cap C = \{p_1, p_2, p_3\}$ are distinct points. For each p_i take the line $L_i = T_{(C,p_i)}$ and define q_i by the condition $L_i \cap C = \{2p_i, q_i\}$. Prove: q_1, q_2, q_3 lie on a line. (You can use question 2.)
  (b) Prove: a line through two (distinct) flexes passes through a third flex. (These nine flexes form the Hesse configuration, see wiki.)
  (c) Let $C \subset \mathbb{P}^2$ be a singular but irreducible cubic. Let $C_{smooth} \subset C$ be the subset of smooth points. Fix some point $0 \in C_{smooth}$ and define the addition of points on C_{smooth} as for smooth cubics. Check that $(C_{smooth}, +, 0)$ is an abelian group.
  (d) Let $C \subset \mathbb{P}^2$ be an irreducible (possibly singular) cubic. Fix two points $0, 0' \in C_{smooth}$ and define the corresponding additions. Define the point $q = \phi(0, 0') \in C_{smooth}$. Prove that the map $(C_{smooth}, +, 0) \xrightarrow{p \to \phi(q,p)} (C_{smooth}, +', 0')$ is an isomorphism of abelian groups.

(4) Let C be a smooth cubic, and suppose $0 \in C$ is a flex.
  (a) Prove: three points $p_1, p_2, p_3 \in C$ lie on a line iff $p_1 + p_2 + p_3 = 0$. (What happens when the points coincide?)
  (b) Prove: a point $p \in C$ is of order 2 in the abelian group iff the line $T_{(C,p)}$ passes through $0 \in C$. Find the points of order two for $C = V(y^2 z - x(x - z)(x - \lambda z))$, $\lambda \neq 0, 1$, $0 = [0 : 1 : 0]$. Compare to hwk10, q.6(c).
  (c) Show that the points of order two on C form the group $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$.
  (d) Prove: in the abelian group C there are precisely 9 elements of order 3, counting 0. (The same as in the complex case, when $C = \mathbb{C}/L$.) These elements correspond to flexes and they form a subgroup, isomorphic to $\mathbb{Z}/(3) \times \mathbb{Z}/(3)$.

(5) (a) Fix some points $pt_1, \dots, pt_4 \in \mathbb{P}^2$. What are the possible dimensions of $V(2, pt_1, \dots, pt_4)$, depending on the location of the points? (Realize each of the possible dimensions by explicit point configurations.)
  (b) The same question for $V(2, pt_1, \dots, pt_n)$, $n \geq 5$ and $V(3, pt_1, \dots, pt_6)$.

(6) ($\Bbbk = \bar{\Bbbk}$) Fix a curve $C_{deg=d} = V(f) \subset \mathbb{P}^2$ and a point $pt_0 = (x_0, y_0, z_0) \notin C$. For any point $pt \in C$ take the line $L = \overline{pt_0, pt}$. How many tangents to C pass through pt_0?
  (a) Prove: $i_{pt}(L, C) > 1$ iff $\{f(pt) = 0 = x_0 \partial_x f + y_0 \partial_y f + z_0 \partial_z f\}$.
  (b) Prove: $i_{pt}(L, C) - 1 = i_{pt}\big(V(f), V(x_0 \partial_x f + y_0 \partial_y f + z_0 \partial_z f)\big)$.
  (c) Suppose C is smooth. Prove: the number of tangent lines to C that pass through $pt_0 \notin C$ is $d(d-1)$.
  Here each line is counted with its multiplicity which is the degree of flex at the tangency point.
  (d) Suppose $C_{deg=d} \subset \mathbb{P}^2$ has a point of multiplicity d. Prove: C is reducible.
  (e) Suppose $C_d \subset \mathbb{P}^2$ has r components, no multiple components, and the points of multiplicities $\{m_i\}$. Prove: $\sum \binom{m_i}{2} \leq \binom{d-1}{2} + r - 1 \leq \binom{d}{2}$. (Hint: we have proved this for irreducible curves. Reduce the question to the irreducible components.)