Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 7

In this homework we assume $\mathbb{k} = \bar{\mathbb{k}}$. All varieties are irreducible. We use only the Zariski topology.

(1) (a) Take a closed affine variety $X \subset \mathbb{k}^n$. (Dis)Prove:
   (i) A map $X \stackrel{\phi}{\to} \mathbb{k}^m$ is continuous iff it is the restriction of a polynomial map $\mathbb{k}^n \to \mathbb{k}^m$.
   (ii) $\mathcal{O}_X(X) = \mathbb{k}[X]$.
   (iii) For $0 \neq f \in \mathbb{k}[X]$ holds: $Y := X \setminus V(f)$ is an affine variety and $\mathcal{O}_Y(Y) = \mathbb{k}[X][f^{-1}]$.
(b) Prove: if $\mathcal{U}' \subset \mathcal{U}$ then $\mathcal{O}_X(\mathcal{U}') \supseteq \mathcal{O}_X(\mathcal{U})$.
(c) Suppose $f \in \mathcal{O}_{(X,pt)}$. Prove: there exists an open neighborhood $pt \in \mathcal{U} \subseteq X$ such that $f \in \mathcal{O}_X(\mathcal{U})$. Prove: $\mathcal{O}_{(X,pt)} = \cup_{pt \in \mathcal{U} \subset X} \mathcal{O}_X(\mathcal{U})$.
(d) Let $a \in \mathcal{O}_X(\mathcal{U})$. Check that a is computable at each point of X and thus defines a function, $\mathcal{U} \stackrel{f_a}{\to} \mathbb{k}$. Prove: if f_a vanishes at each point of some open subset $\varnothing \neq \mathcal{U}' \subseteq \mathcal{U}$ then $a = 0 \in \mathcal{O}_X(\mathcal{U})$, i.e. vanishes as an element of this ring.
Give an example of an affine variety over a non-algebraically closed field $\mathbb{k}$, with $f \in \mathbb{k}[X]$ vanishing at each point of X, but $f \neq 0 \in \mathbb{k}[X]$.
(e) For any variety X and some open subsets $\{\mathcal{U}_i\}$ prove: $\mathcal{O}_X(\cup \mathcal{U}_i) = \cap \mathcal{O}_X(\mathcal{U}_i)$ and $\mathcal{O}_X(\mathcal{U}) = \cap_{pt \in \mathcal{U}} \mathcal{O}_{(X,pt)}$.

(2) (a) Let $X \subset \mathbb{P}^n$ be a closed subvariety and $\mathbb{k}^n = \mathcal{U}_i \subset \mathbb{P}^n$ an affine chart. Prove: the affinization map is an isomorphism of fields, $\mathbb{k}(X) \xrightarrow{\sim} \mathbb{k}(X \cap \mathcal{U}_i)$.
(b) Let X be a variety and $0 \neq f \in \mathbb{k}(X)$. Prove: the set of poles of f is a closed subset in X.
(c) If $X \subseteq \mathbb{k}^n$, $Y \subseteq \mathbb{k}^m$ are closed subvarieties then morphisms $X \to Y$ are exactly the polynomial maps $\mathbb{k}^n \stackrel{\phi}{\to} \mathbb{k}^m$, s.t. $\phi(X) \subseteq Y$. (We have proved this partially in the class.)

(3) (a) Let X be an affine variety and $0 \neq f \in \mathbb{k}[X]$. Define the subset $\mathcal{U}_f := \{f \neq 0\} \subset X$.
Prove: $\mathcal{O}_{\mathcal{U}_f}(\mathcal{U}_f) = \mathbb{k}[X][\frac{1}{f}] := \mathbb{k}[X][y] / (y \cdot f - 1)$ and $\mathcal{U}_f$ is an affine variety.
(b) Let $X \subset \mathbb{P}^n$ be a projective variety and $V_f = \{f = 0\} \subset \mathbb{P}^n$ a hypersurface such that $X \not\subseteq V_f$. Prove: $\mathcal{U}_f = X \setminus (X \cap V_f)$ is an affine variety and identify $\mathbb{k}[\mathcal{U}_f]$. (Hint: in the case $f = x_0$ one has $\mathbb{k}[\mathcal{U}_{x_0}] = \{\frac{\bar{a}}{\bar{x}_0^n} | \, deg(a) = n\}$, here $\bar{a}$, $\bar{x}_0$ are the images of a, x_0 in $\mathbb{k}[X]$.)
(c) Let $\mathbb{k}[Y] \stackrel{\phi}{\to} \mathbb{k}[X]$ be a ring homomorphism. Verify: if $\phi(f)$ is a unit in $\mathbb{k}[X]$ then ϕ extends (uniquely) to $\mathbb{k}[Y][\frac{1}{f}]$. Otherwise ϕ extends uniquely to a homomorphism $\mathbb{k}[Y][\frac{1}{f}] \to \mathbb{k}[X][\frac{1}{f}]$. Give the geometric interpretation.
(d) Prove: $X = \mathbb{k}^2 \setminus \{(0,0)\}$ is a variety but not an affine one. (Hint: what is $\mathcal{O}_X(X)$?)
(e) Prove: any variety is a finite union of affine varieties.

(4) (a) (Dis)Prove:
   (i) The projection $\mathbb{k}^{n+m} \to \mathbb{k}^m$ is a morphism.
   (ii) The projection $\mathbb{P}^{n+m} \setminus \{x_0 = 0 = \cdots = x_n\} \to \mathbb{P}^n$, $(x_0 : \cdots : x_{n+m}) \to (x_0 : \cdots : x_n)$ is a morphism.
   (iii) If $X \subset Y$ be an open/closed subvariety then the inclusion map is a morphism.
   (iv) The composition of morphisms is a morphism.
   (v) The restriction of a morphism to a subvariety is a morphism.
(b) Let $X \stackrel{\phi}{\to} Y$ be a set-theoretic map of varieties. Fix some open covers, $X = \cup \mathcal{U}_i^{(X)}$, $Y = \cup \mathcal{U}_i^{(Y)}$, satisfying: $f(\mathcal{U}_i^{(X)}) \subseteq \mathcal{U}_i^{(Y)}$. Prove: ϕ is a morphism iff all the restrictions $\mathcal{U}_i^{(X)} \xrightarrow{f|_{\mathcal{U}_i^{(X)}}} \mathcal{U}_i^{(Y)}$ are morphisms.
(c) Let $x \in X$ and $y \in Y$ be some points on varieties. Suppose the local rings are isomorphic, $\mathcal{O}_{(X,x)} \xrightarrow{\sim} \mathcal{O}_{(Y,y)}$. Prove: some neighborhoods $x \in \mathcal{U}_X \subseteq X$, $y \in \mathcal{U}_Y \subseteq Y$ are isomorphic, $\mathcal{U}_X \approx \mathcal{U}_Y$.
(d) Let $X \stackrel{\phi}{\to} Y$ be a morphism of affine varieties and $\mathbb{k}[X] \stackrel{\phi^*}{\leftarrow} \mathbb{k}[Y]$ the induced homomorphism of rings. (Dis)Prove: ϕ is injective/surjective iff ϕ^* is surjective/injective. (Check all the possible statements.)
(e) Fix a point on a (smooth) conic, $pt \in C \subset \mathbb{P}^2$ and a line $pt \notin L \subset \mathbb{P}^2$. Define the map $C \setminus \{pt\} \stackrel{\phi}{\to} L$ by sending $p \in C$ to $\overline{p, pt} \cap L$. Is this map surjective? Prove that this map extends to an isomorphism $C \xrightarrow{\sim} \mathbb{P}^1$.