Introduction to Algebraic Curves

201.2.4451. Summer 2019 (D.Kerner)

Homework 8

In this homework we assume $\Bbbk = \bar{\Bbbk}$. All varieties are irreducible. We use only the Zariski topology.

(1) (a) Let $\Bbbk \subset \mathbb{K}$ be a field extension. Prove:
   (i) $x_1, \dots, x_n \in \mathbb{K}$ is a minimal set such that $\mathbb{K}$ is algebraic over $\Bbbk(x_1, \dots, x_n)$ iff $x_1, \dots, x_n \in \mathbb{K}$ is a maximal set of algebraically independent elements of $\mathbb{K}$. Such a set is called: 'a transcendence basis'.
   (ii) Any algebraically independent set can be completed to a transcendence basis.
   (iii) Any set $x_1, \dots, x_n \in \mathbb{K}$ such that $\mathbb{K}$ is algebraic over $\Bbbk(x_1, \dots, x_n)$ contains a transcendence basis.
   (iv) $tr.deg_{\Bbbk}\mathbb{K}$ is the number of elements in any transcendence basis.
(b) Verify: $dim(\Bbbk^n) = n = dim(\mathbb{P}^n)$. $\qquad dim(X \times Y) = dim(X) + dim(Y)$.
(c) Prove: $dim(X) < dim(Y)$ for a closed subvariety $X \subset Y$.
(d) Compute $dim(V(f))$ for a non-constant $f \in \Bbbk[\underline{x}]$.
(e) Let $\Bbbk \subset \mathbb{K}$ be a field extension with $tr.deg_{\Bbbk}(\mathbb{K}) = n$. Suppose $\mathbb{K}$ is generated by N elements, i.e. $\mathbb{K} = \Bbbk(a_1, \dots, a_N)$. Prove: $\mathbb{K} = \Bbbk(X)$ for some closed subvariety $X \subset \Bbbk^N$. Prove: there exists a hypersurface $V \subset \Bbbk^{n+1}$ satisfying: $\Bbbk(V) = \mathbb{K}$. (Geometrically: every variety is birationally equivalent to a hypersurface.)

(2) Fix an algebraic curve $C \subset \Bbbk^n$ and some generators of the defining ideal, $I_C = \langle f_1, \dots, f_N \rangle$. Prove: $pt \in C$ is a smooth point iff the matrix of derivatives of $\{f_i\}$ at pt is of expected rank. (Part of this was proved in the lecture.)

(3) (a) Define a map $\mathbb{P}^1 \stackrel{\nu_d}{\to} \mathbb{P}^d$ by $(x_0 : x_1) \to (x_0^d : x_0^{d-1}x_1 : \dots : x_0 x_1^{d-1} : x_1^d)$, for $d \geq 2$. Prove: ν_d is an injective morphism and its image can be defined as $\{\underline{z} \in \mathbb{P}^d | \ rank \begin{bmatrix} z_0 & \dots & z_{d-1} \\ z_1 & \dots & z_d \end{bmatrix} < 2\}$.
This embedded curve is called *the rational normal curve*. Verify: $\nu_d(\mathbb{P}^1)$ does not lie in any hyperplane of $\mathbb{P}^d$.
(b) Construct an explicit inverse $\nu_d(\mathbb{P}^1) \to \mathbb{P}^1$. (Write explicit formulas.)
(c) Describe/identify $\nu_d(\mathbb{P}^1) \cap \mathcal{U}_i$, for all the standard affine charts, $\mathcal{U}_0, \dots, \mathcal{U}_n \subset \mathbb{P}^n$. Write down $\Bbbk[\nu_d(\mathbb{P}^1) \cap \mathcal{U}_i]$.
(d) (Dis)Prove: $\Bbbk(\nu_d(\mathbb{P}^1)) \xrightarrow{\sim} \Bbbk(\mathbb{P}^1)$; $\Bbbk[\nu_d(\mathbb{P}^1)] \xrightarrow{\sim} \Bbbk[\mathbb{P}^1]$; $\Bbbk[\nu_d(\mathbb{P}^1) \cap \mathcal{U}_i] \xrightarrow{\sim} \Bbbk[\Bbbk^1]$ for some $\mathcal{U}_i$ (for which one?).
(e) Take the projection $\mathbb{P}^d \stackrel{\pi_{0,d}}{\dashrightarrow} \mathbb{P}^1$, $(z_0 : \dots : z_d) \to (z_0 : z_d)$. Prove: this rational map restricts to a morphism $\nu_d(\mathbb{P}^1) \stackrel{\pi_{0,d}}{\to} \mathbb{P}^1$. What is the degree and the ramification data of $\pi_{0,d}$?
(f) Repeat this for the map $\mathbb{P}^d \stackrel{\pi_{i,j}}{\dashrightarrow} \mathbb{P}^1$, $(z_0 : \dots : z_d) \to (z_i : z_j)$, for any $0 \leq i < j \leq d$. What do you get for $j - i = 1$?
(g) Take the projection $\mathbb{P}^d \stackrel{\pi_{0,i,d}}{\dashrightarrow} \mathbb{P}^2$, $(z_0 : \dots : z_d) \to (z_0 : z_i : z_d)$. Prove: this rational map restricts to a birational morphism $\nu_d(\mathbb{P}^1) \stackrel{\pi}{\to} C \subset \mathbb{P}^2$. What is the degree of C? (One way to compute the degree: take a linear form $l(\underline{y})$ on $\mathbb{P}^2$, pull it back to $\Bbbk[\mathbb{P}^1]$ and take the divisor of its zeros.) Write down the defining equation of C and describe its singularities.

(4) (a) Let C be a curve and $pt \in C$. Prove: $Frac(\mathcal{O}_{(C,pt)}) = \Bbbk(C)$.
(b) We have proved: every rational map of a curve extends to all the smooth points. Go over all the details.
(c) Let C, D be projective smooth curves. Prove: $\Bbbk(C) \approx \Bbbk(D)$ iff C, D are isomorphic.
(d) Let C be smooth and D be projective. Prove: dominant morphisms $C \to D$ correspond bijectively to homomorphisms $Hom_{\Bbbk}(\Bbbk(D), \Bbbk(C))$. (A morphism/rational map is called dominant if its image is dense.)
(e) Suppose $C \stackrel{\phi}{\dashrightarrow} D$ is a non-constant rational map of curves. Prove: ϕ is dominant (i.e. $\phi(C)$ is dense in D) and $\Bbbk(C)$ is a finite algebraic extension of $\Bbbk(D)$. Prove: for any $pt \in D$ the fibre $\phi^{-1}_{pt} \subset C$ is a finite set.
(f) Fix two points on curves, $p \in C$, $q \in D$. Prove: any morphism $\mathcal{O}_{(C,p)} \stackrel{\phi}{\to} \mathcal{O}_{(D,q)}$ induces a rational map $D \dashrightarrow C$ with $q \to p$. Prove: if ϕ is an isomorphism then this rational map is birational.
(g) Prove: two smooth projective curves are isomorphic iff exists some $p \in C$, $q \in D$ with $\mathcal{O}_{(D,q)} \approx \mathcal{O}_{(C,p)}$.
(h) Show an infinite collection of pairwise non-isomorphic DVR's.

(5) A rational parametrization of a curve C is a birational morphism $\Bbbk^1 \to C$. Curves that admit a rational parametrization are called rational. Construct explcit rational parametrizations of the following plane curves by setting $y = tx$:
i. $V(x^4 + y^4 - x^2 y)$. $\qquad$ ii. $V(x^3 + y^3 + x^2 - 2y^2)$. $\qquad$ iii. $V(x^4 + y^4 + x^3 + y^3)$.