Introduction to Algebraic Curves

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Homework 9

(1) (a) Assuming the existence of resolution of singularities of curves, prove its uniqueness. (If $\tilde{C}_1, \tilde{C}_2 \to C$ are two resolutions then there exists $\phi: \tilde{C}_1 \xrightarrow{\sim} \tilde{C}_2$ inducing a commutative triangle, and this ϕ is unique.)
(b) Prove: any irreducible singular pane cubic is birational to $\mathbb{P}^1$.
(c) Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d. Suppose $pt \in C$ and $mult_{pt}(C) = d-1$. Prove: C has no other singular points and is birational to $\mathbb{P}^1$.

(2) (a) Let $C = V(x^2 - y^3, y^2 - z^3) \subset \Bbbk^3$. Find a birational morphism $\Bbbk \to C$. Prove that C cannot be embedded into $\Bbbk^2$. (No neighborhood of $(0,0,0) \in C$ is isomorphic to an open subvariety of a plane curve.)
(b) Let $C = V(x_1^2 - x_2^3, x_2^2 - x_3^3, \dots, x_{n-1}^2 - x_n^3) \subset \Bbbk^n$. Prove: C is non-embeddable into $\Bbbk^{n-1}$.
(c) Let $Z \subset \Bbbk^n$ be the union of coordinate axes. Write down $I(C) \subset \Bbbk[\underline{x}]$. Prove: Z cannot be embedded into $\Bbbk^{n-1}$.

(3) (a) Let X be a variety and $\Bbbk \not\ni f \in \Bbbk(X)$. Prove: the field $\Bbbk(f)$ (which is a subfield of $\Bbbk(X)$) is isomorphic to $\Bbbk(\mathbb{P}^1)$. Thus choosing any such f corresponds to an embedding $\Bbbk(\mathbb{P}^1) \xhookrightarrow{i} \Bbbk(X)$.
(b) Prove: any variety X (of positive dimension) admits a dominant rational map $X \dashrightarrow \mathbb{P}^1$.
(c) Prove: any smooth projective curve admits a surjective morphism $X \to \mathbb{P}^1$.

(4) We have proved (algebraically) that every curve is birational to a plane curve. Now we prove: for a projective curve C and $pt \in C$ exists a birational morphism $C \xrightarrow{\phi} C' \subset \mathbb{P}^2$ satisfying: $\phi^{-1}\phi(pt) = pt$.
(a) We can assume $C \subset \mathbb{P}^{n+1}$. Fix the homogeneous coordinates $(t, x_1, \dots, x_n, z)$ such that $pt = (0 : \cdots : 1)$ and $C \cap V(t,z) = \varnothing$ and $C \cap V(t)$ is finite. Then $\Bbbk(C)$ is algebraic over $\Bbbk(u)$ for $u = \frac{t}{z}|_C$.
(b) For each $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Bbbk^n$ define $C \xrightarrow{\phi_{\underline{\lambda}}} \mathbb{P}^2$ by $\phi_{\underline{\lambda}}(t : x_1 : \cdots : x_n : z) = (t : \sum \lambda_i x_i : z)$. Check that $\phi_{\underline{\lambda}}$ is a well defined morphism, $\phi_{\underline{\lambda}}(pt) = (0:0:1)$. Define $C' = \overline{\phi_{\underline{\lambda}}(C)} \subset \mathbb{P}^2$.
(c) Prove that $\underline{\lambda}$ can be chosen such that $\phi_{\underline{\lambda}}$ is a birational morphism and $\phi_{\underline{\lambda}}^{-1}(0:0:1) = pt$.

(5) (a) Suppose two holomorphic maps of compact Riemann surfaces, $X \overset{f,g}{\to} Y$, coincide on an infinite set of points. Prove that they coincide on X.
(b) Let $X \xrightarrow{f} Y$ be a non-constant holomorphic map. Prove that for any $y \in Y$ the set of preimages, $f^{-1}(y)$, is discrete.
(c) Suppose a holomorphic map $X \xrightarrow{f} Y$ is a bijection of sets. Prove: f is holomorphically invertible.
(d) Prove: for any holomorphic map $X \xrightarrow{f} Y$ and any point $x \in X$ there exist local (holomorphic) coordinates on (X, x), for which the map is $f(z) = z^n$. Here n does not depend on the choice of coordinates. (It is called the order/multiplicity of f at x).
(e) Let $X \xrightarrow{f} Y$ a holomorphic map, with X compact. Prove: $ord_x(f) > 1$ only for a finite number of points.
(f) Let f be a non-constant meromorphic function on a compact Riemann surface. Prove: $\sum_{x \in X} ord_x(f) = 0$.

(6) In the lecture we saw how to "plug the holes" in a punctured Riemann surface.
(a) Prove that plugging the holes preserves Hausdorffness and path-connectedness.
(b) Let X be a Riemann surface with punctures, so that the surface $\overline{X}$, obtained by plugging all the holes in X, is a compactification of X. Prove that this compactification is unique, i.e. for any two compact Riemann surfaces $\overline{X}_1 \supset X \subset \overline{X}_2$ holds: $\overline{X}_1 \xrightarrow{\sim} \overline{X}_2$.
(c) Let $C \subset \mathbb{C}^n$ be a singular algebraic curve and let X be the Riemann surface obtained by puncturing the singular points of C and plugging the holes. Take the natural projection $X \xrightarrow{\pi} C$. Fix a point $c \in C$ and some point $x \in \pi^{-1}(c)$. Construct the natural maps $\mathcal{O}_{(\mathbb{C}^n, c)} \to \mathcal{O}_{(C,c)} \to \mathcal{O}_{(X,x)}$. Are they injective/surjective?

(7) Go over all the details of the proof of the formula for the genus of plane curves with ordinary multiple points, $g(C_{d,\{m_i\}})$.