Answers to the Quiz

Question 1

Recall that $log(1+z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_0 = 0$, $a_1 = 1$ and $a_2 = -\frac{1}{2}$. Thus $log(1+z^3) = \sum_{n=0}^{\infty} a_n z^{3n}$, Hence

$$log(1+z^3) - z^3 = -\frac{1}{2}z^6 + \sum_{n\geq 7} b_n z^n,$$

for some coefficients b_n (i.e. the first non-zero coefficient is for z^6. In particular, $log(1+z^3) - z^3 = z^6 g(z)$ for some analytic $g(z)$ so that $g(0) \neq 0$. Thus

$$f(z) = \frac{z^7}{z^6 g(z)} = z\frac{1}{g(z)}.$$

So the order of 0 is 1

Question 2

It is easy to see that $\gamma = \left\{z \in \mathbb{C} : |z| = 2, 0 \leq arg(z) \leq \frac{\pi}{6}\right\}$. We parametrize $\gamma : [0, \pi/6] \to \mathbb{C}$, $\gamma(t) = 2e^{it}$. The formula for $\sqrt[4]{z}$ in the question maps 1 to -1, hence its formula is

$$\sqrt[4]{Re^{i\theta}} = \sqrt[4]{R}e^{i(\theta/4+\pi)}$$

where $0 \leq \theta < \pi$. The function z^3 maps points with $0 \leq arg(z) \leq \frac{\pi}{6}$ to points where $0 \leq arg(z) \leq \frac{\pi}{2}$. In particular, $\sqrt[4]{z^3} = \sqrt[4]{z}^3$ on γ (note that this is *not* true in all the domain of the fourth root). Then

$$\int_\gamma \frac{dz}{\sqrt[4]{z^3}} = -\int_0^{\pi/6} \frac{2ie^{it}}{\sqrt[4]{8}e^{i3t/4+i\pi}}dt = \sqrt[4]{2}i\int_0^{\pi/6} e^{it/4}dt = 4\sqrt[4]{2}e^{it/4}|_{t=0}^{t=\pi/6} = 4\sqrt[4]{2}(e^{i\pi/24} - 1).$$

The minus in the first equality is because we are told to integrate clockwise.

Second Solution

The inverse function derivative theorem states that $f^{-1\prime}(z) = \frac{1}{f'(f^{-1}(z))}$.

In particular, for the function $g(z) = \sqrt[4]{z}$ which is an inverse for z^4. The derivative at z is

$$g'(z) = \frac{1}{4(\sqrt[4]{z})^3}.$$

$$(\sqrt[4]{z})^3 = (\sqrt[4]{Re^{i\theta}})^3 = \sqrt[4]{R^3}e^{i3\theta/4+3\pi} = \sqrt[4]{R^3}e^{3\theta/4+\pi} = \sqrt[4]{z^3}.$$

Note that in general we can't just switch between taking a fourth root and doing a power of 3, and the reason we can do it here, is because $0 \leq \theta \leq \pi/6$.

Thus

$$\int_\gamma \frac{dz}{\sqrt[4]{z^3}} = \int_\gamma \frac{dz}{\sqrt[4]{z}^3} = 4\int_\gamma \frac{dz}{4\sqrt[4]{z}^3} = 4\sqrt[4]{2}(e^{i\pi/24} - 1).$$

Where the second last inequality is due to the fundamental theorem of calculus for the complex numbers.

Question 3

There are many solutions to this question.

Using the fact that f is analytic

We expand f to a power series around 0, and note that from what we saw in class, the series converges in the ball of radius 1. $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

On the one hand $f(\frac{1+i}{3}z) = \sum_{n=0}^{\infty} a_n \left(\frac{1+i}{3}\right)^n z^n$. On the other hand, the coefficients of the series for f are unique, thus for all $n = 0, 1, 2, ...$ we have that

$$a_n = \left(\frac{1+i}{3}\right)^n a_n$$

or

$$(1 - \left(\frac{1+i}{3}\right)^n)a_n = 0.$$

By the fact that $f(0) = 0$ we get that $a_0 = 0$. Furthermore, since since $\left|\left(\frac{1+i}{3}\right)^n\right| < 1$, then $(1-\left(\frac{1+i}{3}\right)^n) \neq 0$ for all $n \geq 1$. Thus for all n, $a_n = 0$, and $f(z)$ is the zero function. In particular $f(\frac{1+i}{3}) = 0$.

Another solution

We restrict f to the *closed* ball $\overline{Ball_{1/2}(0)}$. By the maximum principle, the maximum of f in this restricted domain is at $\partial Ball_{1/2}(0)$. Denote a point where f get's a maximum by z_0. However, since $f(z_0) = f(\frac{1+i}{3}z_0)$, then f get's another maximum in the interior of the ball. Thus by the maximum principle, f is a constant in this restricted domain. By the uniqueness theorem, f is a constant in all $Ball_1(0)$. We know that $f(0) = 0$ thus $f(\frac{1+i}{3}) = 0$.

Another solution

Assume towards contradiction that f is not the zero fucntion and let $ord_0(f) = p \geq 1$. Then $f(z) = z^p g(z)$, for some analytic g so that $g(0) \neq 0$. Now compare $f(z) = f((1+i)z/3)$, to get:

$$(1 + i/3)^p g(0) = g(0).$$

Thus we got $g(0) = 0$ a contradiction.

Another solution: Using continuity only

We prove that f is the constant 0, and in particular $f(\frac{1+i}{3}) = 0$.

Fix $z \in Ball_1(0)$. Denote be $w_n = \left(\frac{1+i}{3}\right)^n z$. Notice that $\left|\left(\frac{1+i}{3}\right)\right| < 1$ thus $\lim_{n\to\infty} w_n = 0$.

From continuity of f, $\lim_{n\to\infty} f(w_n) = f(0)$. On the other hand, we know that $f(w_n) = f(w_0)$ by induction. Thus $f(z) = f(w_0) = f(w_n$ is a constant sequence whose limit is $f(0$, and thus $f(z) = f(0) = 0$ and f is constant.

Question 4

First Solution

Define $g(z) = f(z) - sin(z)$. If we will prove that all its derivatives at 0 are 0, then it will show that the function is the constant 0 (because its expansion to a power series at 0 will be the same as the 0 function).

Note that $g(0) = f(0) - sin(0) = 0$, $g'(0) = f'(0) - cos(0) = 0$. $g''(0) = f''(0) + sin(0)$ and since $f''(0) + f(0) = 0$ then $f''(0) = 0$ and $g''(0) = 0$. $g'''(0) = f'''(0) + cos(0)$ and by deriving the equation $f'''(0) + f'(0) = 0$ we get that $f'''(0) = -f'(0) = -1$ then $g'''(0) = 0$.

To generalize this to every derivative, we use induction on n the number of derivatives.

1. Base case: the cases for $n = 0$ and $n = 1$ were discussed above.

2. Assume that the hypothesis is true for n, we will prove that it is true for $n+2$. Indeed, note that by deriving the equation
$$f''(z) + f(z) = 0$$
we get the equation
$$f^{(n+2)}(z) + f^{(n)}(z) = 0.$$
Indeed, we also know that $sin^{(n+2)}(z) + sin^{(n)}(z) = 0$, thus
$$g^{(n+2)}(0) = f^{(n+2)}(0) - sin^{(n+2)}(0) = -f^{(n)}(0) + sin^{(n)}(0) = -g^{(n)}(0) = 0.$$

Second Solution

We know that $sin(z) = \sum_{n=0}^{\infty} a_n z^n$ in the ball, where
$$a_n = \begin{cases} \frac{(-1)^k}{(2k+1)!} & n = 2k+1, \\ 0 & n = 2k. \end{cases}$$
Denote $f(z) = \sum_{n=0}^{\infty} b_n z^n$. We can write $f(z)$ like this since any analytic function that is defined in $ball_1(0)$ can be written as a power series that converges in the ball.

Thus if we prove that for all $n = 0, 1, 2, ...$, $a_n = b_n$, then $f(z) = sin(z)$. We saw in class that we can derive the series one by one, thus
$$f''(z) = \sum_{n=0} (n+2)(n+1) b_{n+2} z^n.$$
We know that $f''(z) + f(z) = 0$, hence
$$\sum_{n=0} (n+2)(n+1) b_{n+2} z^n + \sum_{n=0} b_n z^n = 0.$$
Since if a function is 0 then all its coefficients in the power series are 0 then for all n we get that
$$(n+2)(n+1) b_{n+2} + b_n = 0.$$

Now we prove that
$$b_n = a_n$$
by induction on n:

1. Base Cases: When $n = 0, 1$ then $b_0 = f(0) = 0 = a_0$ and $b_1 = f'(0) = 1 = a_1$ by our assumptions.

2. Assume that $a_n = b_n$ and prove that $a_{n+2} = b_{n+2}$: Indeed, we know that
$$b_{n+2} + (n+2)(n+1) b_n = 0,$$
or
$$b_{n+2} = (-1) \frac{1}{(n+2)(n+1)} b_n = (-1) \frac{1}{(n+2)(n+1)} a_n,$$
where the last equality is by the induction hypothesis. If $n = 2k$ is even then $b_{n+2} = 0 = a_{n+2}$. If $n = 2k+1$ is odd then
$$b_{n+2} = (-1) \frac{1}{(2k+3)(2k+1)} \frac{(-1)^k}{(2k+1)!} = a_{n+2},$$
and the claim follows.

Third Solution

We know that $f(z) + f''(z) = 0$ in the ball. In particular, this is true in the real interval $(-1, 1)$. From a theorem in ODE there is a unique solution to this equation where $f(0) = 0, f'(0) = 1$ and this is $sin(z)$. Thus $f(z) = sin(z)$ in the real interval. This interval has a sequence with a cluster point (for example $z_n = \frac{1}{n}$ whose limit is 0). Since $f(\frac{1}{n}) = sin(\frac{1}{n})$, then by the uniqueness theorem, $f(z) = sin(z)$ on the whole domain $Ball_1(0)$.

Question 5

Yes - f must be analytic.

If f is the constant 0 then obviously f is analytic. So assume otherwise.

Notice that $f(z) = \frac{f(z)^3}{f(z)^2}$ is defined on all points where $f(z) \neq 0$ and is analytic there since both $f(z)^3, f(z)^2$ are analytic. $f(z)^3$ and $f(z)^2$ are not constant, then their zeros are isolated and have finite order. If we prove that for every $z_0 \in \mathbb{C}$ so that $f(z_0) = 0$ we have that $ord_{z_0}(f^2) < ord_{z_0}(f^3)$, then in particular $\frac{f(z)^3}{f(z)^2}$ is analytic on z_0 (when it is defined to be 0), thus f is analytic.

Indeed, if $f(z_0) = 0$ then $f(z)^6 = f(z)^3 \cdot f(z)^3 = f(z)^2 \cdot f(z)^2 \cdot f(z)^2$, thus $0 < ord_{z_0}(f^6) = 2ord_{z_0}(f^3) = 3ord_{z_0}(f^2)$, and thus $ord_{z_0}(f^2) < ord_{z_0}(f^3)$.