MOED A: SOLUTIONS

1. Question 1:

Rewrite the function as

$$f(z) = \frac{5z-1}{z^2-z-2} = \frac{5z-1}{(z+1)(z-2)} = \frac{2}{z+1} + \frac{3}{z-2} = \frac{2}{w} + \frac{3}{w}\Big(\frac{1}{1-\frac{3}{w}}\Big),$$

where $w = z+1$. Therefore, for $|w| > 3$, we have

$$f(z) = \frac{2}{w} + \frac{3}{w}\sum_{n=0}^{\infty}\frac{3^n}{w^n} = \frac{5}{z+1} + \sum_{n=2}^{\infty}\frac{3^n}{(z+1)^n}$$

where the sum converges for every $|z+1| = |w| > 3$ and diverges whenever $|z+1| \leq 3$; In particular the series converges for $3 < |z+1| < 4$.

2. Question 2:

The function

$$f(z) = \frac{e^{1/z}}{\sin(1/z)}$$

is analytic in $\mathbb{C}$ except for the singularity points $z = 0$ and $z = 1/\pi k$ for every $k \in \mathbb{Z}$. For every $k \in \mathbb{Z}$ we have

$$|1/\pi k - 1/2| \leq 1/\pi + 1/2 < 1,$$

so all the singularities of f lies inside $Ball_1(1/2)$, therefore we have

$$\int_{|z-1/2|=1} f(z)dz = -2\pi i Res_{z=\infty}(f) = -2\pi i Res_{z=0}\Big(\frac{e^z}{z^2\sin z}\Big),$$

where $z = 0$ is a pole of the function of order 3, so using the Taylor series of $e^z, \sin z$:

$$Res_{z=0}\left(\frac{e^z}{z^2\sin z}\right) = Res_{z=0}\left(\frac{e^z}{z^3(1-z^2/6)}\right) = Res_{z=0}\left(\frac{e^z}{z^3}\sum_{n=0}^{2}(z^2/6)^n\right)$$
$$= Res_{z-0}\left(\frac{(1+z+z^2/2)(1+z^2/6)}{z^3}\right) = 1/6 + 1/2 = 2/3$$

so the integral is equal to $-4\pi i/3$.

3. Question 3:

Same as in the midterm.

4. Question 4:

• The function $f_1(z) = ze^{-i\pi/4}$ maps $\{z = x + iy : x > y\}$ onto $\{z : Im(z) < 0\}$;

• We look for a Mobius function $f_2(z) = \frac{az+b}{cz+d}$ which maps $\{z : Im(z) < 0\}$ onto $Ball_1(0)$: say the function maps $z = 0$ to -1 and also $z = \infty$ to 1, thus $b/d = -1$ and $a/c = 1$, thus $f_2(z) = \frac{az+b}{az-b}$. Let f maps $z = 1$ to i, so $a + b = i(a - b)$ and fix $a = 1$, then $b = i$. Therefore the Mobius function

$$f_2(z) = \frac{z+i}{z-i}$$

maps $\{z : Im(z) = 0\}$ onto $\{z : |z| = 1\}$, while $f_2(-i) = 0$, so f_2 maps $\{z : Im(z) < 0\}$ onto $Ball_1(0)$.

• The function $f_3(z) = 2z$ maps $Ball_1(0)$ onto $Ball_2(0)$.

• Therefore the function

$$f(z) := f_3(f_2(f_1(z))) = 2\left(\frac{ze^{-i\pi/4} + i}{ze^{-i\pi/4} - i}\right)$$

maps $\{z = x + iy : x > y\}$ onto $Ball_2(0)$, while f is conformal mapping as f_1, f_2 and f_3 are conformal mappings.

5. Question 5:

Let $f(z) = 2/z^6 - 3/z^3 + 1$ and $\mathcal{U} = \{z : |z| > 1\}$. Since $z \mapsto \overline{z}$ is a bijective map from $\mathcal{U}$ to $\mathcal{U}$, (as $z \in \mathcal{U}$ iff $\overline{z} \in \mathcal{U}$),

$$M = \sup_{z\in\mathcal{U}} |f(\overline{z})| = \sup_{z\in\mathcal{U}} |f(z)|.$$

Let $p(z) := 2z^6 - 3z^3 + 1$. As $z \mapsto 1/z$ is a bijective map from $\mathcal{U}$ to $Ball_1(0) \setminus \{0\}$, we have

$$M = \sup_{z\in\mathcal{U}} |p(1/z)| = \sup_{z\in Ball_1(0)\setminus\{0\}} |p(z)|.$$

The function $p(z)$ is analytic in $\mathbb{C}$ and non-constant, therefore by the maximum principle we have

$$M = \sup_{|z|<1} \left|2z^6 - 3z^3 + 1\right| = \max_{|z|=1} |2z^6 - 3z^3 + 1|.$$

It is easily seen that $|p(z)| \le 2|z|^6+3|z|^3+1 = 6$ for every $|z| = 1$, while $|p(-1)| = 6$, therefore $M = 6$.

6. Question 6:

• **First solution:** Assume that the statement does not hold. Suppose there exists $R > 0$ such that for every $n_0 \in \mathbb{N}$, there exists $n \ge n_0$ and $z_n \in Ball_R(0)$ such that $p_n(z_n) = 0$. Therefore, there exists $r > 0$ and a sequences $n_1 < n_2 < \ldots \in \mathbb{N}$ and $z_{n_k} \in Ball_R(0)$ such that $p_{n_k}(z_{n_k}) = 0$, for all $k \ge 1$. Without loss of generality suppose that z_{n_k} converges (the sequence is in $Ball_R(0)$ hence it has a converges sub-sequence), say $z_{n_k} \to w$.

Let $\varepsilon > 0$. We know that the sequence $\{p_n(z)\}$ converges uniformly to the function e^z in $Ball_{R+1}(0)$, therefore there exists $N \in \mathbb{N}$ such that for every $n > N$ and for every $z \in Ball_{R+1}(0)$ we have

$$|p_n(z) - e^z| < \varepsilon/2,$$

in particular there exists $K_1 \in \mathbb{N}$ such that

$$|e^{z_{n_k}}| = |p_{n_k}(z_{n_k}) - e^{z_{n_k}}| < \varepsilon/2$$

for every $k > K_1$. On the other hand, as the function e^z is continuous and $z_{n_k} \to w$, there exists $K_2 \in \mathbb{N}$ such that

$$|e^{z_{n_k}} - e^w| < \varepsilon/2$$

for every $k > K_2$. Thus

$$|e^w| \leq |e^w - e^{z_{n_k}}| + |e^{z_{n_k}}| < \varepsilon$$

for every $k > \max\{K_1, K_2\}$, which implies that $e^w = 0$ and this is a contradiction.

• Second solution: Let $f(z) = e^z$. For every $R > 0$, the function f does not vanish in $\overline{Ball_R(0)}$ and being a continuous function on a compact set, there exists $\varepsilon > 0$ such that $|f(z)| > \varepsilon$ for every $z \in \overline{Ball_R(0)}$. As $p_n(z)$ converges uniformly to e^z in $\overline{Ball_R(0)}$, there exists $n_0 \in \mathbb{N}$ such that $|p_n(z) - e^z| < \varepsilon$ for every $n \geq n_0$ and $z \in \overline{Ball_R(0)}$. Thus $|p_n(z)| \geq ||p_n(z) - e^z| - |e^z|| = |e^z| - |p_n(z) - e^z| > 0$, which implies that $p_n(z) \neq 0$ for every $z \in \overline{Ball_R(0)}$ and $n \geq n_0$.