MOED B: SOLUTIONS

Question 1

The only singularity of the function is at $z = 0$, however $0 \notin \mathcal{U}$, so the function is analytic in $\mathcal{U}$ and then it follows from Cauchy's theorem that the integral is equal to 0.

Question 2

Let us find a Moebius mapping φ which maps $\partial Ball_1(0)$ to a line and also $\mathbb{R}$ to a line, with both lines going through 0!

As $-1, 1 \in \mathbb{R} \cap \partial Ball_1(0)$ we require that $\varphi(-1) = \infty$ and $\varphi(1) = 0$, so we have

$$\varphi(z) = \frac{az - a}{cz + c} = \frac{a}{c}\frac{z-1}{z+1}.$$

Next, let us fix $\varphi(i) = 1$, i.e., $\frac{a}{c} = .. = -i$, which implies that $\varphi(z) = i\frac{1-z}{z+1}$ and we know that φ maps $\partial Ball_1(0)$ to $\mathbb{R}$; since $\varphi(0) = i$, we have

$$\varphi(Ball_1(0)) = \{z : Im(z) > 0\}.$$

Now—using the points $-1, 1, 0$— we also know that φ maps $\mathbb{R}$ into $i\mathbb{R}$, but $\varphi(i) = 1$ so

$$\varphi(\{z : Im(z) > 0\}) = \{z : Re(z) > 0\}.$$

Therefore,

$$\varphi(\{z : Im(z) > 0, |z| < 1\}) = \{z : Re(z) > 0,\ Im(z) > 0\}.$$

Question 3

Let $f(z) = 1/p(z)$, where $p(z) = 5z^9 - 3z^2 + 1$. For every $|z| = 1$:

$$|-3z^2 + 1| \leq 3 + 1 = 4 < 5 = |5z^9|,$$

thus by Rouche's theorem: the number of zeros of $p(z)$ in $Ball_1(0)$ is equal to the number of zeros of $5z^9$ in $Ball_1(0)$ which is 9. On the other hand $p(z)$ has 9 zeros in $\mathbb{C}$, therefore $p(z) \neq 0$ for every $|z| \geq 1$. We get that f has all of its singularity points in $Ball_1(0)$, so from the theorem of residues:

$$\int_{|z|=1} f(z)dz = -2\pi i Res_{z=\infty}(f) = 2\pi i Res_{z=0}\Big(\frac{z^7}{5 - 3z^7 + z^9}\Big) = 0$$

since the last function is analytic at $z = 0$.

Question 4

From the assumption we get that $f(\mathbb{C}) \subseteq \mathcal{U}$, where
$$\mathcal{U} := \{z = x + iy : y < e^x\}.$$
If f is non-constant, then (as $f \in \mathcal{O}(\mathbb{C})$) its image $f(\mathbb{C})$ is dense in $\mathbb{C}$ and therefore $\mathcal{U}$ is dense in $\mathbb{C}$; this is clearly not true, as for example
$$f(\mathbb{C}) \cap Ball_1(-2 + 2i) = \emptyset.$$
Therefore f must be constant.

Common mistake: "The function $\frac{v(x,y)}{e^{u(x,y)}}$ is not entire and bounded, hence constant". This function can not be entire (at any sense), as it is real!

Question 5

Let $p(z) = a_0 + \ldots + a_n z^n$ and define the polynomial $q(z) := a_n + \ldots + a_0 z^n$. Notice that for every $z \neq 0$ we can write $q(z) = z^n p(1/z)$. For every $|z| = 1$, we have $|q(z)| = |p(1/z)| \leq M$, since $|1/z| = 1$. From the maximum principle we get that also for every $|z| \leq 1 : |q(z)| \leq M$. In particular, for every $|z| \geq 1$, we have
$$|q(1/z)| \leq M \Longrightarrow \left|\frac{p(z)}{z^n}\right| \leq M \Longrightarrow |p(z)| \leq M|z|^n.$$

Common mistake: "as $|p(z)| \leq M$ for $|z| = 1$, we have
$$|p(z)| = |\sum_{k=0}^{n} a_k z^k| \leq \sum_{k=0}^{n} |a_k||z|^n = \sum_{k=0}^{n} |a_k| \leq M."$$
This is not true at all! "This is like saying: If $x < 0$, then $x < 2 < 0$."

Question 6

Let $Log(z)$ be the main branch ("Anaf Rashi") of logarithm in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The given function $f(z) = \log(z)$ is then must be of the form
$$f(z) = Log(z) + 2\pi k i$$
for some $k \in \mathbb{Z}$. As $f(1) = 8\pi i$, we get that $8\pi i = 2\pi k i$ and thus $k = 4$, so
$$f(z) = Log(z) + 8\pi i, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}.$$
For every $z = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, here $r > 0$ and $-\pi < \theta < \pi$, we have
$$|f(z)| = |\ln(r) + i(\theta + 8\pi)| \geq |\theta + 8\pi| \geq 7\pi.$$
On the other hand, for every $z \in Ball_1(2)$, $|z - 2| < 1$ and hence
$$|z^2 + 4| = |(z-2)^2 + 4z| \leq 1 + 4|z - 2 + 2| \leq 1 + 4(1 + 2) = 13 < 7\pi = |f(z)|,$$
which means that $z^2 + 4 \neq f(z)$ for every $z \in Ball_1(2)$. The answer is 0 solutions.

Another solution: Write $z = re^{i\theta} \in Ball_1(2)$, then $-\pi < \theta \leq \pi$ and $r \leq 1$. If $f(z) = z^2 + 4$, then $\ln(r) + i(\theta + 8\pi) = r^2 e^{2\theta i} + 4$, which implies that
$$\ln(r) - 4 = r^2 \cos(2\theta) \text{ and } \theta + 8\pi = r^2 \sin(2\theta).$$
Square these equations and add them up, so
$$(7\pi)^2 \leq (\theta + 8\pi)^2 \leq (\ln(r) - 4)^2 + (\theta + 8\pi)^2 = r^4 \leq 1,$$
that is a contradiction. So there are no solutions in $Ball_1(2)$.

Common mistake: "$|\log(z)| \geq 8\pi$ for every $z \in Ball_1(2)$". This is not true.