MOED C: SOLUTIONS

Question 1

No. Here is a counter example: let $f(z) = z^2$, then $F(x, y) = (Re(f), Im(f)) = (x^2 - y^2, 2xy)$. Thus, $(Re(f))_y = -2y \neq 2y = (Im(f))_x$ in $Ball_1(0)$, which implies that F is not a local conservative vector field in $Ball_1(0)$.

Notice that $(-Re(f), Im(f))$ is indeed a local conservative vector field, due to the Cauchy–Riemann equations.

Question 2

Let $g(z) = \frac{1}{1+z^2} \in Ball_1(0)$ and $z_n = 1/n$ for $n \in \mathbb{N}$. From the assumption, $g(z_n) = f'(z_n)$ for every $n \in \mathbb{N}$, meaning the two functions f' and g, both in $\mathcal{O}(Ball_1(0))$, coincide on a sequence of points, and this sequence converges inside $Ball_1(0)$, which imply by the uniqueness theorem that $f'(z) = g(z)$ for every $z \in Ball_1(0)$, i.e.,

$$f'(z) = \frac{1}{1+z^2} \Longrightarrow f(z) = \arctan(z) + C,$$

but $f(0) = 3\pi$, thus $f(z) = \arctan(z) + 3\pi$ and $f(1/2) = \arctan(1/2) + 3\pi$.

Question 3

Let $f(z) = \frac{\sin(z)-\cos(z)}{(z-\pi/4)^3}$ and $g(z) = \sin(z) - \cos(z)$. Let γ be a closed and simple path in $\mathcal{U} := Ball_1(0) \setminus \{\pi/4\}$ and distinguish the two cases: (1) if the point $\pi/4$ does not belong to the interior of γ, then $\int_\gamma f(z)dz = 0$ by the Cauchy-Goursat theorem. (2) if the point $\pi/4$ belongs to the interior of γ, then by the Cauchy formula (for $n = 2$) we get

$$\int_\gamma f(z)dz = \int_\gamma \frac{g(z)}{(z-\pi/4)^3}dz = \frac{2\pi i}{2!} g^{(2)}(\pi/4) = 0,$$

as $g^{(2)}(z) = -\sin(z) + \cos(z)$. An **alternative** way to compute the integral is to notice that $g(z) = \sqrt{2}\sin(z - \pi/4) = \sqrt{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}(z-\pi/4)^{2n+1}$ and thus the Laurent series of f around $z = \pi/4$ is

$$f(z) = \sum_{n=0}^\infty \frac{\sqrt{2}(-1)^n}{(2n+1)!}(z-\pi/4)^{2n-2},$$

hence by the residues theorem, $\int_\gamma f(z)dz = 0$.

Therefore, $\int_\gamma f(z)dz = 0$ for every closed and simple path γ in $\mathcal{U}$, which implies by the Morera's theorem that there exists $g \in \mathcal{O}(\mathcal{U})$ such that $g'(z) = f(z)$ for every $z \in \mathcal{U}$. In particular, g is analytic in $\mathcal{U}$.

Typical mistakes:
• The Taylor series expansion of $\sin(z)$ (and similarly $\cos(z)$) around $z = \pi/4$ is **not**

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z - \pi/4)^{2n+1},$$

meaning when you consider Taylor series around different points, the coefficients change as well.
• The point $z = \pi/4$ is not a pole of order 3 of the function $f(z)$, as it is a zero of order 3 of the numerator but also a zero of order 1 of the denominator.

Question 4

From the assumption that $ord_0(f) \geq n$, there exists $g \in \mathcal{O}(Ball_1(0))$ such that $f(z) = z^n g(z)$ for every $z \in \overline{Ball_1(0)}$. Then for every $|z| = 1$ we have

$$|g(z)| = |z^n g(z)| = |f(z)| \leq 1,$$

which implies using the maximum principle that $|g(z)| \leq 1$ for every $|z| \leq 1$. Therefore, for every $|z| \leq 1$: $|f(z)| = |z^n g(z)| \leq |z|^n$.

Typical mistakes:
• The argument: "$f(z) = z^n g(z)$, therefore $|g(z)| \leq 1$" is not an immediate, as one can see in our proof, and requires explanation.
• The argument: "$f(0) = 0$ and $|f(z)| \leq 1$, therefore by the Schwartz lemma $|f(z)| \leq |z|^n$" is not immediate (and not even true without the extra assumption that $ord_0(f) \geq n$).

Question 5

For every $0 \leq t \leq 2\pi$, let $z = e^{it}$, thus $2\cos(t) = e^{it} + e^{-it} = z + 1/z$ and $\frac{dz}{iz} = dt$. Therefore,

$$I = \int_0^{2\pi} \frac{dt}{1 - 2a\cos(t) + a^2} = \int_{|z|=1} \frac{dz}{iz(1 - a(z + 1/z) + a^2)} = -i \int_{|z|=1} \frac{dz}{(z-a)(1-az)}.$$

Notice that $|a| < 1$ and $|1/a| > 1$, therefore using the Cauchy formula for the function $f(z) = \frac{1}{1-az}$, we have

$$I = -i(2\pi i)f(a) = \frac{2\pi}{1-a^2}.$$

Question 6

Let $g(z) = zf(z) \in \mathcal{O}(\mathcal{U})$. By the assumption $|g(z)| = |zf(z)| \leq |z|\ln(1/|z|)$ for every $z \in \mathcal{U}$, therefore $\lim_{z\to 0} |g(z)| = 0$, as this is an immediate limit of real valued functions from Hedva 1, e.g. using l'Hopital's rule. Thus, $\lim_{z\to 0} g(z) = 0$ which means that $z = 0$ is a removable ("Slika") singularity point of $g(z)$. As g is analytic and $g(0) = 0$, it holds that $g(z) = zh(z)$ where $h \in \mathcal{O}(Ball_1(0))$. Then $g(z) = zf(z) = zh(z)$ in $Ball_1(0)$, implies that $h(z) = f(z)$ in $Ball_1(0) \setminus \{0\}$, however h is analytic in $Ball_1(0)$ which implies that f is analytic in $Ball_1(0)$ as well.