Solutions to some questions of Moed.A, Geometric Calculus 1, 06.02.2020

Question 2.b. Take a path $[0,1] \xrightarrow{\gamma} C$, with $\gamma(0) = (0,1)$, $\gamma(1) = (1,0)$. Denote the image of γ by C_γ.

Solution 1. Consider the set $S = \{d | \ C_\gamma \cap \{x+y=d\} \neq \varnothing\}$. (All the values of d for which the line intersects the curve.) Note: $S \ni \{1\}$.

As C_γ is compact, there exists $S \ni d_0 = sup(S)$. Assume $d_0 > 1$, let $(x_0, y_0) \in C_\gamma \cap \{x+y=d_0\}$. Then C is tangent to $\{x+y=d_0\}$ at (x_0,y_0). (Indeed, $grad(f)$ does not vanish on C, and if $grad_{(x_0,y_0)} f \not\sim (1,1)$ then d_0 cannot be the supremum)

If $d_0 = 1$ then take $S \ni d_0 = inf(S)$.

If $inf(S) = sup(S) = 1$ then C_γ is a segment inside $\{x+y=1\}$.

Solution 2. Project C_γ onto the line $\{y=x\}$. Note that the points $(1,0)$, $(0,1)$ are sent to the same point $(\frac{1}{2},\frac{1}{2})$. If all the points are sent to this point then γ is the straight segment from $(1,0)$ to $(0,1)$.

Otherwise, the image of C_γ (under this projection) has the minimum and maximum on the line $\{y=x\}$. (The image is a compact subset.) And at least one of these maximum/minimum comes from the tangency to $\{x+y=d\}$.

Solution 3. Define the map $C \to S^1 \in \mathbb{R}^2$ by $C \ni x \to \frac{grad(f)}{||grad(f)||} \in S^1$. (This is well defined as f' does not vanish at any point.) This map is continuous, and sends $(1,0) \to (0,1)$, $(0,1) \to (1,0)$. The image is path-connected, thus some point of C is sent to $\pm(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

Question 3.b.

Solution 1. Fix some $\vec{v} \in \mathbb{R}^3$. We are looking for the min/max of the function $g(\underline{x}) = \underline{x} \cdot \underline{v}$ on S. The min/max of g on S exist, as g is continuous and S is compact. This is the min/max problem with the restriction $f(\underline{x}) = 0$. Therefore the conditional extrema of g satisfy: $grad(f) \sim grad(g) = \underline{v}$. Thus, if the min/max are different, we get at least two points of S where $grad(f)||\underline{v}$.

If the min/max coincide, then S lies inside the plane $\underline{x} \cdot \underline{v} = const$. But then at any point of S we have:

- either $grad(f) \neq 0$ and thus S is locally a part of the plane, hence $grad(f)||\underline{v}$ for infinity of points;
- or $grad(f) = 0$ and thus trivially $grad(f)||\underline{v}$ for at least two points.

Solution 2. Rotate $\mathbb{R}^3$ to assume $\underline{v} = (0,0,1)$. Consider the function z on S. This is a continuous function on a compact set, thus it has the extrema. At each extremum holds:

- either $grad(f) \neq 0$, and then $grad(f)||(0,0,1)$;
- or $grad(f) = 0$, thus $grad(f)||(0,0,1)$.

If the min/max of z are realized at distinct points of S then we get the needed two points. If the min/max are realized at the same point then S sits inside the plane $z = z_0$. Then argue as above.

Question 4.b.

The condition $\frac{y}{2} \le z \le y$ implies $y \ge 0$, thus $|y| = y$. We pass to the iterated integral, and then to polar coordinates:

$$\int_{-1}^{1}\Big(\iint\limits_{\substack{0\le y^2+z^2\le x^{\frac{2}{3}}\\ \frac{y}{2}\le z\le y}} ye^{x^2}dydz\Big)dx = \int_{-1}^{1}\Big(\iint\limits_{\substack{0\le r\le |x|^{\frac{1}{3}}\\ arctan(\frac{1}{2})\le\phi\le\frac{\pi}{4}}} r^2cos(\phi)drd\phi\Big)e^{x^2}dx = (\frac{1}{\sqrt{2}} - sin(arctan(\frac{1}{2}))\int_{-1}^{1}\frac{|x|}{3}e^{x^2}dx =$$

$$= 2\frac{(\frac{1}{\sqrt{2}} - sin(arctan(\frac{1}{2}))}{3}\int_{0}^{1}xe^{x^2}dx = \frac{(\frac{1}{\sqrt{2}} - sin(arctan(\frac{1}{2}))}{3}\cdot(e^1-1).$$

One can also compute $sin(arctan(\frac{1}{2}))$, e.g., $sin^2 = \frac{sin^2}{sin^2+cos^2} = \frac{tan^2}{tan^2+1}$, thus $sin(arctan(\frac{1}{2})) = \frac{1}{\sqrt{5}}$.