Solutions to some questions of Moed.B, Geometric Calculus 1, 27.02.2020

Question 2.b.

(1) (A remark) As A is symmetric, it can be diagonalized by an orthogonal transformation, $A \to UAU^t$, $UU^t = 1\!\!1$. This transformation preserves the sphere S^{n-1} and the eigenvalues of A. Thus we can assume $f(x) = \sum \lambda_i x_i^2$, here $\{\lambda_i\}$ are the eigenvalues of A. Thus the absolute min/max value of f on S^{n-1} coincides with the smallest/largest eigenvalue of A, and is achieved at the corresponding eigenvector.

(2) (The general solution) The critical points of f under the constraint $g(x) := \sum x_i^2 - 1 = 0$ satisfy: $grad(f) \sim grad(g)$.

By the direct check, $grad(f)|_x = 2A \cdot x$ and $grad(g)|_x = 2x$. Thus any critical point satisfies: $A \cdot x \sim x$, $x \neq 0$, i.e. is an eigenvector of A. Let $\{\vec{v}_i\}$ be the eigenvectors, with their eigenvalues $\{\lambda_i\}$. We can choose them ortho-normalized. Then $f(\vec{v}_i) = \lambda_i$. Thus the absolute minimum of f is $min\{\lambda_i\}$, the absolute maximum of f is $max\{\lambda_i\}$.

To classify the other critical points suppose $\lambda_i < \lambda_j$. Restrict the function to the subset

$$\mathbb{R}^2 \supset Span_{\mathbb{R}}(\vec{v}_i, \vec{v}_j) \cap S^{n-1} = S^1.$$

We get $f(t_i, t_j) = \lambda_i t_i^2 + \lambda_j t_j^2$, where $t_i^2 + t_j^2 = 1$. This has a local minimum at $t_j = 0$ and a local maximum at $t_i = 0$.

Thus, if $\lambda_i < \lambda_j < \lambda_k$ then the critical point $\vec{v}_j$ is a saddle point. (It is a local minimum when restricted to one smooth curve and a local maximum for the other.)

Question 3.b.

Solution 1. Suppose $dim(V) = 3$, then the matrix $[f'|_{x_0}, g_1'|_{x_0}, g_2'|_{x_0}]$ is non-degenerate. Consider the map $\mathbb{R}^n \overset{f,g_1,g_2}{\to} \mathbb{R}^3$. By the open mapping theorem its image is locally open at the point $(f(x_0), g_1(x_0), g_2(x_0))$. But then in any neighborhood of x_0 there are points p, q such that

$$g_1(p) = g_1(x_0), \quad g_2(p) = g_2(x_0), \quad f(p) > f(x_0), \quad \text{and } g_1(q) = g_1(x_0), \quad g_2(q) = g_2(x_0), \quad f(q) < f(x_0).$$

Thus x_0 cannot be a local minimum of f on $\partial\mathcal{U}$.

Solution 2. Suppose $dim(V) = 3$, then the matrix $[g_1'|_{x_0}, g_2'|_{x_0}, f'|_{x_0}]$ is non-degenerate. Then, by a local coordinate change, the map $\mathbb{R}^n \overset{g_1,g_2,f}{\to} \mathbb{R}^3$ can be brought to the form (x_1, x_2, x_3). But then x_0 cannot be a local minimum of f on $\partial\mathcal{U}$.

Solution 3. Suppose x_0 is a local minimum of f on $\partial\mathcal{U}$. Thus x_0 is a local minimum under one of the constraints: $\{g_1 = 0\}$ or $\{g_2 = 0\}$ or $\{g_1 = 0 = g_2\}$. In either case, Lagrange's theorem gives: the vectors $f'|_{x_0}, g_1'|_{x_0}, g_2'|_{x_0}$ are linearly dependent. Thus dim $Span(f'|_{x_0}, g_1'|_{x_0}, g_2'|_{x_0}) \leq 2$.

Question 4.a. The function $Mat_{n\times n}(\mathbb{R}) \overset{det}{\to} \mathbb{R}$ is infinitely differentiable, being a polynomial. Thus its derivative can be computed in one of the (equivalent) ways.

Solution 1. Fix the (natural) basis of $Mat_{n\times n}(\mathbb{R})$ corresponding to the standard basis of $\mathbb{R}^{n^2}$. We compute the partial derivatives of $det(\dots)$. To compute $\frac{\partial det(\dots)}{\partial a_{ij}}$ one expands $det(\dots)$ according to i'th row or j'th column. Then one gets: $\frac{\partial det(\dots)}{\partial a_{ij}} = (A^\vee)_{ji}$. Thus $det(\ )'|_A(\Delta) = \sum_{ij} (A^\vee)_{ij} \Delta_{ij} = trace(A^\vee \cdot \Delta)$.

Solution 2. The linear map $det()'|_A \in Hom(Mat_{n\times n}(\mathbb{R}), \mathbb{R})$ is determined by its action on some basis of the vector space $Mat_{n\times n}(\mathbb{R})$. Take the basis of elementary matrices, E_{ij}. Then

$$det()'|_A(E_{ij}) = \lim \frac{det(A + tE_{ij}) - det(A)}{t}.$$

And again expand $det(A + tE_{ij})$ according to i'th row or j'th column, to get: $det(\ )'|_A(E_{ij}) = (A^\vee)_{ji}$.