Geometric Calculus 1, 201.1.1031

Homework 0

Fall 2019 (D.Kerner)

Linear geometry and matrices The material below should be well known. However ...

(1) The standard inner product on $\mathbb{R}^n$ is defined by $\langle\vec{v},\vec{u}\rangle := \sum v_i u_i$, $||\vec{v}|| = \sqrt{\sum v_i^2}$.

(a) Prove the Cauchy-Schwartz inequality, $|\langle\vec{v},\vec{u}\rangle| \le ||\vec{v}|| \cdot ||\vec{u}||$, and the triangle inequality, $||\vec{v}+\vec{u}|| \le ||\vec{v}|| + ||\vec{u}||$.

(b) The angle between two (non-zero) vectors in $\mathbb{R}^n$ is defined by $\angle(\vec{v},\vec{u}) := arccos\frac{\langle\vec{v},\vec{u}\rangle}{||\vec{v}||\cdot||\vec{u}||} \in [0,\pi]$. Check that this is well defined, and depends only on the directions of the vectors. (Namely, $\angle(\lambda\vec{v},\delta\vec{u}) = \angle(\vec{v},\vec{u})$, for any $\lambda,\delta > 0$.) Prove Pythagorean theorem: $\vec{v}\perp\vec{u}$ iff $||\vec{v}||^2 + ||\vec{u}||^2 = ||\vec{v}+\vec{u}||^2$.

(c) The shift by $\vec{v}$ (or "parallel translation") is the map $T_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n$, $\vec{x} \to \vec{x}+\vec{v}$. Is this a linear operator in the sense of linear algebra?
(Dis)Prove: i. $T_{\vec{v}} \circ T_{\vec{u}} = T_{\vec{v}+\vec{u}}$ ii. $T_{\vec{v}}$ preserves the angles and the distances

(2) (a) A plane $L \subseteq \mathbb{R}^n$ is the set of solutions of a (given) linear system of equations, $\{A\vec{x} = \vec{b}\}$. For which systems are the planes vector subspaces of $\mathbb{R}^n$? Is the system of equations defined uniquely by L?
(Dis)Prove: i. Any shift of any plane is a plane. ii. Any L is a shift of a vector subspace $V_L \subset \mathbb{R}^n$.
iii. V_L defined uniquely by L. iv. The shift from L to V_L is defined uniquely by L.
iv. For any $\vec{v} \neq 0$ holds: $T_{\vec{v}}(L)$ is a plane and $T_{\vec{v}}(L) \cap L = \varnothing$.

(b) The dimension of a plane is $dim(L) := dim(V_L)$. A line is a plane of dimension 1.
The codimension of a plane is $codim(L) := n - dim(L)$. A hyperplane is a plane of codimenions 1. Can a line be a hyperplane?
A coordinate plane is a plane of the form $\cap_{i\in I}\{x_i = 0\}$, for a subset $I \subset \{1,\ldots,n\}$. Compute the number of coordinate planes of codimension k in $\mathbb{R}^n$.

(c) (i) Prove: $dim(T_{\vec{v}}(L)) = dim(L)$. (What is the system of equations defining $T_{\vec{v}}(L)$?)
(ii) Prove: if $L_1 \cap L_2 \neq \mathbb{R}^n$ then $codim(L_1 \cap L_2) \le codim(L_1) + codim(L_2)$. When does the equality hold?
(iii) Prove: $codim(L)$ =the minimal number of equations needed to define L.
(iv) Prove that the images and preimages of planes under linear transformations are planes.
Warning: Most of the statements above might seem obvious (as this is what we see in $\mathbb{R}^2$, $\mathbb{R}^3$). However, in $\mathbb{R}^{n>3}$ we need rigorous proofs.

(d) Present a plane as the shifted vector space, $L = T_{\vec{v}}(V)$. The normal space to the plane, $V^\perp$, is the orthogonal complement of $V \subseteq \mathbb{R}^n$. Check: the normal space does not depend on the choice of $\vec{v}$. Prove: the normal to the hyperplane $\{\sum a_i x_i = b\}$ is one-dimensional, $Span(a_1,\ldots,a_n)$.

(3) Denote by $Mat_{m\times n}(\mathbb{R})$ the vector space of $m\times n$-matrices. The elementary matrices, $\{E_{ij}\}$, form the standard basis. The inner product is $\langle A,B\rangle := tr(AB^t)$, the norm is $||A|| := \sqrt{tr(AB^t)}$.

(a) What are the matrices orthogonal to $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$? (Describe the corresponding subspace)

(b) Define the map $Mat_{m\times n}(\mathbb{R}) \xrightarrow{\phi} \mathbb{R}^{mn}$ by $\sum a_{ij}E_{ij} \to \{a_{ij}\}$. Check that this is an isomorphism of vector spaces; it maps $\{E_{ij}\}$ to the standard basis of $\mathbb{R}^{mn}$; and the inner product/norm on $Mat_{m\times n}(\mathbb{R})$ are induced from those on $\mathbb{R}^{mn}$, i.e. $\langle A,B\rangle = \langle\phi(A),\phi(B)\rangle_{\mathbb{R}^{mn}}$, $||A|| = ||\phi(A)||_{\mathbb{R}^{mn}}$.

(c) For which points of $\mathbb{R}^3$ holds: $rank\begin{bmatrix}x^2 & y^3 & z^2\\ y & x^2 & z\end{bmatrix} = 1$? (Hint: no need to arrive at the canonical form, rather check the vanishing of all the 2×2 minors).

(d) Recall that any homogeneous quadratic polynomial, $p(\underline{x}) = \sum c_{ij}x_i x_j$, can be (uniquely) presented in the form $\underline{x}^t A\underline{x}$, where $\underline{x} \in Mat_{n\times 1}(\mathbb{R})$ and $A = A^t \in Mat_{n\times n}(\mathbb{R})$. Express A via $\{c_{ij}\}$. Prove: $p(\underline{x}) > 0$ for any $0 \neq \underline{x} \in \mathbb{R}^n$ iff all the eigenvalues of A are positive. (The road: recall that any symmetric matrix is orthogonally diagonalizable, $A = U\cdot Diag\cdot U^t$, $UU^t = \mathbb{1}$. Thus it suffices to check: $\underline{x}^t\cdot Diag\cdot\underline{x} > 0$ for any $0 \neq \underline{x} \in \mathbb{R}^n$.)

(e) Prove: the function $f(x,y,z) = 3x^2 - 5xy + 7y^2 + z^4$ is bounded from below on $\mathbb{R}^3$.

(f) Fix a set of points $\{\underline{x}^{(1)},\ldots,\underline{x}^{(k)}\}$ in $\mathbb{R}^n$. (Here $\underline{x}^{(i)} = (x_1^{(i)},\ldots,x_n^{(i)})$.) Associate to them the matrix $A := \left[\underline{x}^{(1)}\ldots\underline{x}^{(k)}\right] \in Mat_{m\times k}(\mathbb{R})$. Prove: the points lie all on one line through the origin iff the rank of this matrix is at most 1.
What is the condition for the points for the points to lie all on a plane of dimension m? When is this plane unique?

(4) Curves and domains in $\mathbb{R}^2$. Much of the material below should be known in some form. However ...

(a) Prove that through every point of the region $\{y < x^2\} \subset \mathbb{R}^2$ pass exactly two lines tangent to the curve $\{y = x^2\}$.
(b) Draw the domains in $\mathbb{R}^2$: i. $\{|x|+|y| \le 1\}$ ii. $\{|2x-y|+|2y-x| \le 1\}$ iii. $\{-1 \le xy \le 1,\ -1 \le x-y \le 1\}$.
(c) Draw the graph of $f(x) = |x|^\alpha$. Here distinguish the cases: $\alpha < 0$, $\alpha = 0$, $0 < \alpha < 1$, $\alpha = 1$, $\alpha > 1$.
(d) Draw the curve $\{|x|^\alpha + |y|^\alpha = 1\}$, $\alpha > 0$. (Hint: it is enough to consider the case $x, y > 0$. And here we have the graph of the function $\sqrt[\alpha]{1-x^\alpha}$.) What do we get for $\alpha = 1$, $\alpha = 2$, $\alpha = 100$?
(e) Draw the curves in $\mathbb{R}^2$ defined by the following equations. In each case identify the geometric meaning of a, b. Some of the curves are related by rotations/scaling of $\mathbb{R}^2$. Identify these pairs and the corresponding linear transformations.
i. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ii. $\frac{(x+y)^2}{a^2} + \frac{(x-y)^2}{b^2} = 1$ iii. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
iv. $\frac{(x+y)^2}{a^2} - \frac{(x-y)^2}{b^2} = 1$ v. $\{\frac{x^2}{a^2} = \frac{y^2}{b^2}\}$ vi. $\{yx = 1\}$.
(f) (Canonical forms of plane curves of degree 2) Consider the general quadratic polynomial, $p(x, y) = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}$ and the corresponding curve $\{p(x, y) = 0\} \subset \mathbb{R}^2$. Assume that at least one of the coefficients $a_{2,0}$, $a_{1,1}$, $a_{0,2}$ is non-zero. Prove: by rotations and shifts of $\mathbb{R}^2$ this curve can be brought to one (and only one) of the following cases: a parabola $\{y = ax^2\}$, an ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, a hyperbola $\{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\}$, two parallel lines, two intersecting lines, a double line ($x^2 = 0$), a point, an empty set. (Instructions: present the quadratic part of $p(x, y)$ via the matrix, as in question 3.d. This matrix is of rank 1 or 2. Apply a rotation to $\mathbb{R}^2$ to diagonalize this matrix. Now get rid of the remaning linear part of $p(x, y)$.)
(g) Is the subset $\{2x^2 + 7xy + 4y^2 - 10x - 15y = 30\}$ bounded?
(h) Write the equation of the curve obtained by the full (2π) rotation of the point $(2, 3)$ around the point $(1, 2) \in \mathbb{R}^2$.
(i) Draw the following curves (defined in the polar coordinates on $\mathbb{R}^2$)
i. $\{r = cos(\phi)\}$ ii. $\{r = |sin(6\phi)|\}$ iii. $\{r = \phi,\ \phi \in [0, \infty)\}$ iv. $\{r = cos^2(\phi)\}$.

(5) Curves and surfaces in $\mathbb{R}^3$.
(a) Define the curve $C \subset \mathbb{R}^3$ as the image of the map $\mathbb{R} \stackrel{\phi}{\to} \mathbb{R}^3$, $\phi(t) = (t, t^2, t^3)$. Write the equations of the projections of C onto the planes (x, y), (y, z), (x, z). Prove: no three distinct points of C lie on one line. Prove: no four distinct points of C lie on one plane. (Use question 3.f, above)
(b) Draw/imagine/identify the following surfaces in $\mathbb{R}^3$:
i. $\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2\}$ ii. $\{\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1\}$ iii. $\{x^2 + y^2 = 1\}$
iv. $\{z = x^2 + y^2\}$ v. $\{y = x^2\}$ vi. $\{x^2 - y^2 = 1\}$ (Warning: the last three are not curves!)
(c) Write the equation of the surface obtained by the full (2π) rotation of the curve $\{y^2 + z^2 = R^2,\ x = 0\} \subset \mathbb{R}^3$ around the z -axis.
(d) Write the equation of the surface obtained by the full (2π) rotation of the curve $\{z^2 - y^2 = 1,\ x = 0\} \subset \mathbb{R}^3$ around the z -axis. Draw the surface. (It is called: a hyperboloid with two sheets)
(e) Write the equation of the surface obtained by the full (2π) rotation of the curve $\{(y-r)^2+z^2 = R^2,\ x = 0\} \subset \mathbb{R}^3$ around the z -axis. Here $0 < r < R$. Draw the surface. (It is called: a torus)

(6) A bit of Calculus 1 and 2
(a) The functions below are not defined on the whole $\mathbb{R}$. To which (maximal) subset of $\mathbb{R}$ they can be extended in a differentiable way? i. $f(x) = e^{-\frac{1}{|x|}}$ ii. $f(x) = (1 + sin(x))^{cotan(2x)}$.
(b) Prove that the function $\mathbb{R} \stackrel{f}{\to} \mathbb{R}$, $f(x) = x^2 \cdot arctan(x)$ is invertible. Is the inverse function continuous/differentiable?
(c) Let $f(x) = |cos(x)|^{\frac{1}{|sin(x)|}} + x \cdot ln(1 + \frac{1}{x^4})$. Is f uniformly continuous on $(-1, 0)$? On $(1, 100) \setminus \pi\mathbb{Z}$? On $(-\infty, \infty) \setminus \pi\mathbb{Z}$?
(d) Take a function $\mathbb{R} \stackrel{f}{\to} \mathbb{R}$. (Dis)Prove:
(i) If $|f(x) - f(y)| \le C|x - y|^\alpha$, for some $\alpha > 1$, $C > 0$ and any points $x, y \in \mathbb{R}$, then $f = const.$
(ii) If $\lim\limits_{x\to+\infty} f'(x) = 0$ then for any constant a holds: $\lim\limits_{x\to+\infty} (f(x + a) - f(x)) = 0$. Is f necessarily bounded?
(iii) If f is infinitely differentiable then for any c exist x_1, x_2 such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$.
(e) Compute: i. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{2n-1} \frac{k^\alpha}{n^{\alpha+1}}$ ii. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k^2 arctan \frac{k}{n}}{n^3}$ iii. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{ln(n+k)-ln(n)}{n}$.
(Hint: use the Riemann sums)
(f) Compute: i. $\lim\limits_{x\to+\infty} \frac{\int_0^x e^{t^2} dt}{e^{x^2}}$ ii. $\lim\limits_{x\to+\infty} \frac{\int_0^x (arctan(t))^2 dt}{\sqrt{x^2+1}}$ iii. $\lim\limits_{x\to 0} \frac{\int_{sin(sin(x))}^{sin(x)} e^{t^2} dt}{x^2}$ iv. $\lim\limits_{x\to 0^+} \frac{\int_0^{tan(x)} t \cdot sin(at)dt}{x - sin(x)}$.
(g) Prove that the Dirichlet function, $\chi(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ is not integrable on any interval of $\mathbb{R}$.
(h) Prove that the Thomae function, $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q},\ p \in \mathbb{Z}, q \in \mathbb{Z}_{>0},\ coprime \\ 0, & x \notin \mathbb{Q} \end{cases}$, is integrable on any finite interval. Compute $\int_a^b f(x)dx$.