Geometric Calculus 1, 201.1.1031

Homework 1. (Not for submission)

Fall 2019 (D.Kerner)

(1) (a) Take several points, $(x_1, y_1), \dots, (x_k, y_k)$ in $\mathbb{R}^2$. Prove: they all lie on one line iff the matrix $\begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_k \\ y_1 & \dots & y_k \end{bmatrix}$ is of rank< 3.

(b) Extend this to: "the points $\underline{x}^{(1)}, \dots, \underline{x}^{(k)}$ in $\mathbb{R}^n$ lie on one hyperplane iff ... "

(2) Draw/describe the sections of the following surfaces (in $\mathbb{R}^3$) by the planes $\{x = x_0\}$, $\{y = y_0\}$, $\{z = z_0\}$. Use this to draw/imagine the surfaces.

(a) i. $\{z = \frac{x^2}{a^2} + \frac{y^2}{b^2}\}$, called "elliptic paraboloid". ii. $\{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}\}$, called "hyperbolic paraboloid" (saddle). iii. $\{z = xy\}$ iv. $\{z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}\}$, called "elliptic cone". v. $\{\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1\}$, called "two-sheeted hyperboloid". vi. $\{\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\}$, called "one-sheeted hyperboloid".

(b) i. $\{z = sin(x)\}$ ii. $\{z = sin(x^2 + y^2)\}$ iii. $\{sin(x^2 + y^2 + z^2) = 1\}$.

(3) Draw/describe the following subsets of $\mathbb{R}^3$: i. $\Big\{1 - x^2 - y^2 \ge z \ge x^2 + y^2 - 1\Big\}$ ii. $\Big\{0 \le z \le \frac{1}{xy},\ |x| + |y| \le 1\Big\}$ iii. $\Big\{x^2 + y^2 + z^2 \le 1,\ (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \le 1\Big\}$ iv. $\Big\{\sqrt{x^2 + y^2} \le z \le \sqrt{1 - x^2 - y^2}\Big\}$ v. $\Big\{z^2 \le x^2 + y^2 + 1,\ x^2 + y^2 \le 1\Big\}$.

(4) A level curve of a function $\mathbb{R}^2 \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}$ is the subset $\{f(x, y) = c\} \subset \mathbb{R}^2$, for some $c \in \mathbb{R}$.

(a) Check that different level curves never intersect and that they cover the whole $\mathscr{D}_f$.

(b) Draw all the level curves for the graphs of the following functions. Using them try to draw/describe the graphs of the functions. i. $f(x, y) = \frac{1}{x^2 + 2y^2}$ ii. $f(x, y) = \frac{x}{y}$ iii. $f(x, y) = \frac{xy}{x^2 + y^2}$ (At some points the level curves become very dense. Can you interpret this in terms of $\lim\limits_{(x,y)\to\cdots} (\cdots)$?)

iv. $f(x, y) = |x| + |y| - |x + y|$ v. $f(x, y) = \frac{y^2 + x^2 - 1}{x^2 + 4}$.

(5) (a) Compute the length of the longest diagonal in the box $[0, 1]^n \subset \mathbb{R}^n$, i.e. the length of the edge from $(0, \dots, 0)$ to $(1, \dots, 1)$. (Note: the distances in a perfectly bounded body cannot be bounded uniformly in n.)

(b) Denote by r_n the radius of the largest closed ball that lies inside $[0, 1]^n$. Denote by R_n the radius of the smallest closed ball that contains $[0, 1]^n$. Compute $\lim_{n\to\infty} \frac{R_n}{r_n}$.

(6) (a) Take a function $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$. Check: the image of the map $\mathbb{R}^n \ni \underline{x} \to (\underline{x}, \underline{f}(\underline{x})) \in \mathbb{R}^m$ is the graph of $\underline{f}$.

(b) Let $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ be a linear map, i.e. $f(\underline{x} + \underline{y}) = f(\underline{x}) + f(\underline{y})$, and $f(\lambda \cdot \underline{x}) = \lambda \cdot f(\underline{x})$, for any $\lambda \in \mathbb{R}$, $\underline{x}, \underline{y} \in \mathbb{R}^n$.

(i) Prove: there exists (and unique) a matrix, $A \in Mat_{n\times n}(\mathbb{R})$, such that $f(\underline{x}) = A \cdot \underline{x}$, for any $\underline{x} \in \mathbb{R}^n$.

(ii) Prove: $|f(\underline{x})| \le |A| \cdot |\underline{x}|$, for any $\underline{x} \in \mathbb{R}^n$. (The norm $|A|$ is as in homework 0.)

(7) (a) Find the interior, the closure and the boundary of the following sets. Check whether the sets are open/closed.

i. $\{y^2 = z^2 + x^2 - 1\} \subset \mathbb{R}^3$ ii. $\prod(a_i, b_i] \subset \mathbb{R}^n$ iii. $Ball_r(\underline{a}) \subset \mathbb{R}^n$ iv. $\overline{Ball_r(\underline{a})} \subset \mathbb{R}^n$ v. $\mathbb{Q}^n \subset \mathbb{R}^n$ vi. $S^{n-1} := \{\underline{x}|\ ||\underline{x}|| = 1\} \subset \mathbb{R}^n$ vii. $\cup_{n,m\in\mathbb{N}} \overline{Ball_{\frac{1}{2^{m+n}}}(\frac{1}{m}, \frac{1}{n})} \subset \mathbb{R}^2$

(b) Denote by π_x, π_y the projections of $\mathbb{R}^2$ onto the coordinate axes. Construct a subset $S \subset \mathbb{R}^2$ such that the restrictions $\pi_x|_S$, $\pi_y|_S$ are injective, and $\partial(S) = \mathbb{R}^2$.

(8) (Dis)Prove the following statements. If a statement happens to be false, can you correct it?

(a) $S \subset \mathbb{R}^n$ is open iff $\mathbb{R}^n \setminus S$ is closed.

(b) If $S_1 \subset \mathbb{R}^n$, $S_2 \subset \mathbb{R}^m$ are open/closed then so is $S_1 \times S_2 \subset \mathbb{R}^n \times \mathbb{R}^m$.

(c) Any union of open/closed subsets of $\mathbb{R}^n$ is open/closed.

(d) Any intersection of open/closed subsets of $\mathbb{R}^n$ is open/closed.

(e) $\varnothing \ne S \subset \mathbb{R}^n$ is open iff $S \cap \partial(S) = \varnothing$ iff $int(\partial(S)) = \varnothing$.

(f) An open/closed subset of $\mathbb{R}^n$ is at most a countable union of open/closed balls.

(g) For any subset $S \subset \mathbb{R}^n$ holds: i. $S \subseteq \overline{int(S)}$ ii. $S \supseteq int(\overline{S})$ iii. $\partial\overline{S} = \partial S$ iv. $\partial(int(S)) = \partial S$.