# Geometric Calculus 1, 201.1.1031

## Homework 10

Fall 2019 (D.Kerner)

(1) (a) Verify that $||L||_{op}:=\sup_{v\neq 0}\frac{||L(v)||}{||v||}$ defines a norm on $Hom(\mathbb{R}^n,\mathbb{R}^m)$.
Verify: $||L_1\circ L_2||_{op}\leq ||L_1||_{op}\cdot||L_2||_{op}$.

(b) Take the presentation matrix $[L]$ for the standard bases of $\mathbb{R}^n$, $\mathbb{R}^m$. Find the min/max of $f(\underline{x})=||[L]\cdot\underline{x}||$ on $S^{n-1}$. Express $||L||_{op}$ via the eigenvalues of $[L]^t\cdot[L]$.

(c) We have defined the differentiability/$C^k$ of functions $\mathbb{R}^n\supseteq\mathscr{D}_f\stackrel{f}{\to}\mathbb{R}^m$ for the Euclidean norms on $\mathbb{R}^n,\mathbb{R}^m$. Prove: $f$ is differentiable/$C^k$ for some norms $||*||_{\mathbb{R}^n}$, $||*||_{\mathbb{R}^m}$ iff it is differentiable/$C^k$ for any other norms $\widetilde{||*||}_{\mathbb{R}^n}$, $\widetilde{||*||}_{\mathbb{R}^m}$.

(d) According to these norms one has $||f'|_p||_{op}$ and $\widetilde{||f'|_p||}_{op}$. Prove: there exist $0<c<C<\infty$ such that (for any $p\in\mathscr{D}_f$): $c\cdot\widetilde{||f'|_p||}_{op}\leq ||f'|_p||_{op}\leq C\cdot\widetilde{||f'|_p||}_{op}$.

(2) (a) Let $\mathbb{R}^n\supseteq\mathscr{D}_f\stackrel{f}{\to}\mathbb{R}^m$ be $C^1$. Prove: for any compact subset $K\subset\mathscr{D}_f$ exists $c>0$ such that $||f(\underline{y})-f(\underline{x})||\leq c\cdot||\underline{x}-\underline{y}||$ for any $\underline{x},\underline{y}\in K$. Can $C^1$ be weakened to differentiability?

(b) Let $\mathbb{R}^n\supseteq\mathscr{D}_f\stackrel{f}{\to}\mathbb{R}^m$ be $C^1$ at $p\in\mathscr{D}_f$. Prove: for any $\epsilon>0$ exists $\delta>0$ such that for any $\underline{x},\underline{y}\in Ball_\delta(p)$ holds: $||f(\underline{y})-f(\underline{x})||\leq||f'|_p(\underline{y}-\underline{x})||+\epsilon||\underline{y}-\underline{x}||$.

(3) Let $[0,1]\stackrel{\gamma}{\to}\mathbb{R}^n$ be $C^0[0,1]\cap C^1(0,1)$, injective and such that $\frac{d\gamma}{dt}\neq 0$ for any $t\in[0,1]$. (In this case the image, $C=\gamma[0,1]$, is called a regular curve.)

(a) Prove: for each point $p\in C$ there exists a choice of local coordinates in which the curve is (locally) a line. (Such a coordinate choice is called "a rectification of the curve".)

(b) Prove: there exists $\delta>0$ such that $\gamma(t)+\vec{v}\notin C$ for any $t\in[0,1]$ and any $0\neq|\vec{v}|<\delta$ which is not tangent to $C$, i.e. not parallel to $\frac{d\gamma}{dt}$.

(4) (a) Check that the function $Mat_{n\times n}(\mathbb{R})\stackrel{det}{\to}\mathbb{R}$ is $C^\infty$. Prove: $(det)'|_A(\Delta)=trace(A^\vee\cdot\Delta)$. (Here $A^\vee$ is the adjugate of $A$).

(b) Let $\Sigma^{n\times n}_r\subseteq Mat_{n\times n}(\mathbb{R})$ be the subsets of matrices of $rank\leq r$, here $0\leq r\leq n$.

(i) Verify: $\{\mathbb{O}\}=\Sigma^{n\times n}_0\subsetneq\Sigma^{n\times n}_1\subsetneq\cdots\subsetneq\Sigma^{n\times n}_n=Mat_{n\times n}(\mathbb{R})$.

(ii) Present $\Sigma^{2\times 2}_{rank\leq 1}$ as a quadratic cone in $\mathbb{R}^4$. What are its smooth points?

(iii) Are $\{\Sigma^{n\times n}_r\}_r$ closed? Are they path-connected?

(iv) Prove: $\Sigma^{n\times n}_{n-1}$ is a smooth hypersurface outside of $\Sigma^{n\times n}_{n-2}$.

(c) Compute $f'|_0$ for the function $f(t)=det(\mathbb{1}+t\cdot A)$ .

(d) Check that the function $GL(n,\mathbb{R})\ni A\to A^{-1}\in GL(n,\mathbb{R})$ is $C^\infty$. Compute the first derivative.

(e) Prove that $O(n,\mathbb{R})$ is compact. Find $min/max$ of the function $trace$ on $O(n,\mathbb{R})$. (Hint: no need to do the machinery of extrema under constraints)

(5) Define the map $Mat_{n\times n}(\mathbb{R})\stackrel{exp}{\to}Mat_{n\times n}(\mathbb{R})$ by $exp(A)=\sum\limits_{j=0}^{\infty}\frac{A^j}{j!}$ and the map $Mat_{n\times n}(\mathbb{R})\supset Ball_1(\mathbb{1})\stackrel{ln}{\to}Mat_{n\times n}(\mathbb{R})$ by $ln(\mathbb{1}-A)=-\sum\limits_{j=1}^{\infty}\frac{A^j}{j}$. (Convention: $A^0=\mathbb{1}$)

(a) Compute $exp(A)$, $ln(\mathbb{1}-A)$ for a diagonalizable matrix.

(b) Prove: these power series converge absolutely, and the convergence is uniform on compact subsets. You can use $||A\cdot B||\leq||A||\cdot||B||$. (One can show that these maps are $C^\infty$.)

(c) Prove: if $A,B$ commute then $exp(A+B)=exp(A)exp(B)$ and $ln(A\cdot B)=ln(A)+ln(B)$.

(d) Fix some $A\in Mat_{n\times n}(\mathbb{R})$ and define $\mathbb{R}^1\stackrel{\gamma}{\to}Mat_{n\times n}(\mathbb{R})$, by $\gamma(t)=exp(t\cdot A)$. Compute $\frac{d\gamma}{dt}$.