Geometric Calculus 1, 201.1.1031

Homework 11

Fall 2019 (D.Kerner)

Below $Box := \prod_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$, $\{b_i > a_i\}$. We have defined $vol_n \prod[a_i, b_i] := \prod(b_i - a_i)$.

(1) The following statements have been claimed/partially proved in the lecture. Prove them.

(a) $vol_n(Box) = \sum_{\square_\alpha \in P} vol_n(\square_\alpha)$ for any partition.

(b) For any partitions P_1, P_2 of Box there exists a partition P' satisfying: $P' \leq P_1, P_2$.

(c) If $P' \leq P$ then $L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P)$. Therefore $\sup_{P_1} L(f,P_1) \leq \inf_{P_2} U(f,P_2)$.

(d) For any finite cover by boxes, $X \subset \cup \square_\alpha$, there exists a partition P of X such that each box of P lies in some $\square_\alpha$.

(e) If the sets $\{S_i\}_{i\in\mathbb{N}}$ are all of measure zero then $\mu_n(\cup S_i) = 0$.

(f) Prove: $\mu_n(X) = 0$ iff for every $\epsilon > 0$ exists a countable cover by *open* boxes, $X \subset \cup \square_\alpha$, with $\sum vol_n(\square_\alpha) < \epsilon$.

(g) Let $X \subseteq \cup S_\alpha \subset \mathbb{R}^n$ be a cover of a compact set by sets satisfying: $\overline{int(S_\alpha)} = \overline{S_\alpha}$. Suppose $\{S_\alpha\}$ admits locally a finite subcover. (i.e. for any $x \in X$ exists $Ball_\delta(x) \cap X$ that can be covered by a finite number of $\{S_\alpha\}$) Prove: $\cup S_\alpha$ contains a finite subcover.

(h) Let $\mathbb{R}^n \supseteq \mathscr{D} \xrightarrow{f} \mathbb{R}$, with $\mathscr{D}$ compact, and assume $o(f,x) < \epsilon$ for any $x \in \mathscr{D}$. Then exists $\delta > 0$ such that $o(f, Ball_\delta(\underline{x})) < \epsilon$ for each $\underline{x} \in \mathscr{D}$.

(2) (a) Let p_k be a convergent sequence of points in $\mathbb{R}^n$. Prove: $vol_n(\cup p_k) = 0$.

(b) (Dis)prove: i. If $X \subset \mathbb{R}^n$ is an unbounded set then $vol_n(X) \neq 0$; $\mu_n(X) \neq 0$.
ii. If $vol_n(X) = 0$ then $vol_n(\partial X) = 0$. iii. If $\mu_n(X) = 0$ then $\mu_n(\partial X) = 0$.
iv. If $X \subset \mathbb{R}^1$ is closed and $\mu_1(X) = 0$ then X has vol_1 (possibly zero).
v. If $vol_n(S_1) = 0$ and S_2 is bounded then $vol_{n+m}(S_1 \times S_2) = 0$.
vi. If $\mu_n(S) = 0$ then $\mu_{n+m}(S \times \mathbb{R}^m) = 0$. vii. If $\mu_n(X) = 0$ then $\overline{X} \neq \mathbb{R}^n$.

(c) Suppose $[a,b] \xrightarrow{f} \mathbb{R}$ is monotonic. Prove: f is continuous off a set of measure zero. (Thus integrable.)

(d) Let $\mathbb{R}^n \supset \mathscr{D} \xrightarrow{f} \mathbb{R}^m$ be continuous bounded function, and $\mathscr{D}$ a bounded set admitting vol_n. Prove: $vol_{n+m}(\Gamma_f) = 0$. (The case $m = 1$ was proved in the class.)

(e) Let $X = \{\underline{x} | \ f(\underline{x}) = 0\} \subset \mathbb{R}^n$, where f is C^1, and $f' \neq 0$ except for a subset of zero volume. Prove: $vol_n(X) = 0$.
Give many examples of curves in $\mathbb{R}^2$, surfaces in $\mathbb{R}^3$, etc., of $vol_n = 0$.

(3) (a) We have defined $\int_{\mathscr{D}} f d^n\underline{x} = \int_{Box} f \cdot \mathbb{I}_{\mathscr{D}} d^n\underline{x}$. Prove: this does not depend on the choice of Box.

(b) (The functions are assumed integrable unless stated otherwise) Prove:
i. f is integrable iff for any $\epsilon > 0$ exists a parition P for which $U(f,P) - L(f,P) < \epsilon$.
ii. If f, g are integrable then $f + g$ and $f \cdot g$ are integrable.
iii. If $f \leq g$ then $\int_{\mathscr{D}} f d^n\underline{x} \leq \int_{\mathscr{D}} g d^n\underline{x}$. iv. $|\int_{\mathscr{D}} f d^n\underline{x}| \leq \int_{\mathscr{D}} |f| d^n\underline{x}$.
v. If $vol_n \mathscr{D} = 0$ then $\int_{\mathscr{D}} f d^n\underline{x} = 0$. vi. If $f = g$ off a subset of volume 0 then $\int_{\mathscr{D}} f d^n\underline{x} = \int_{\mathscr{D}} g d^n\underline{x}$.

(c) Prove: for any $X \subset \mathbb{R}^n$ that admits volume and any $\epsilon > 0$ there exists a compact subset $Y \subseteq X$ such that $vol_n(X) - vol_n(Y) < \epsilon$.

(d) If $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ admit volume then also $X \times Y$ and $vol_{n+m}(X \times Y) = vol_n(X) vol_m(Y)$.

(4) For $a < b$ verify: $\int_b^a f dx = -\int_{[a,b]} f dx = -\int_{[b,a]} f dx$. (We have defined $\int_{\mathscr{D}} f$ in the unoriented manner)