Geometric Calculus 1, 201.1.1031

Homework 12

Fall 2019 (D.Kerner)

Below $Box := \prod_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$, $\{b_i > a_i\}$. We have defined $vol_n \prod[a_i, b_i] := \prod(b_i - a_i)$.

(1) Construct a set $S = \cup_{k=1}^{\infty}(a_k, b_k) \subset [0, 1]$ such that $Q \cap (0, 1) \subset S$ and $\sum(b_k - a_k) < 1$. Prove:
   (a) $[0, 1] \setminus S$ is a compact set, with empty interior, not of measure zero.
   (b) S and $[0, 1] \setminus S$ do not admit vol_1.

(2) (a) Given $X \subseteq Box \subset \mathbb{R}^n$ and a partion P of Box define $\lceil X \rceil_P$, $\lfloor X \rfloor_P$ as the union of boxes of P the intersect/lie inside X. Prove: X admits vol_n iff $\inf_P \lceil X \rceil_P = \sup_P \lfloor X \rfloor_P = vol_n(X)$.
   (b) Prove: if $\mathbb{R}^n \supset S_1 \cup S_2 \xrightarrow{f} \mathbb{R}$ is integrable and S_1, S_2, $S_1 \cap S_2$ admit volume then $\int_{S_1 \cup S_2} f d^n \underline{x} = \int_{S_1} f d^n \underline{x} + \int_{S_2} f d^n \underline{x} - \int_{S_1 \cap S_2} f d^n \underline{x}$.
   (c) Suppose $\mathbb{R}^n \supset \mathscr{D} \xrightarrow{f} \mathbb{R}_{\geq 0}$ is integrable ($\mathscr{D}$ is bounded). Prove: $\int\limits_{\mathscr{D}} f d^n \underline{x} = 0$ iff $f = 0$ off a set of measure 0.
   (d) Let $\mathbb{R}^n \supset \mathscr{D}_{bounded} \xrightarrow{f} \mathbb{R}$ and $\mathscr{D} = \cup S_\alpha$ is a countable cover by sets admitting vol_n. (Dis)prove: f is integrable iff each $f|_{S_\alpha}$ is integrable.

(3) (a) (Integral mean value theorem) Let $\mathbb{R}^n \supset \mathscr{D} \xrightarrow{f} \mathbb{R}$ integrable and continuous, with $\mathscr{D}$ path connected. Prove: $\int\limits_{\mathscr{D}} f d^n \underline{x} = f(\underline{x}_0) \cdot vol_n(\mathscr{D})$, for some $\underline{x}_0 \in \mathscr{D}$.
   (b) Let $S \subset \mathbb{R}^n$, $vol_n(S) > 1$. Prove: exists $\underline{x}, \underline{y} \in S$ such that $\underline{x} \equiv \underline{y}\ mod(\mathbb{Z}^n)$.

(4) (a) Let $\mathbb{R}^{n_{\underline{x}}}_{\underline{x}} \times \mathbb{R}^{n_{\underline{y}}}_{\underline{y}} \supset \mathscr{D} \xrightarrow{f} \mathbb{R}$ be integrable. Take the projection $\mathscr{D} \xrightarrow{\pi} \mathbb{R}^{n_{\underline{x}}}_{\underline{x}}$.
      i. Obtain (from Fubini on Box): $\int\limits_{\mathscr{D}} f(\underline{x}, \underline{y}) d^{n_{\underline{x}}}\underline{x} d^{n_{\underline{y}}}\underline{y} = \int\limits_{\pi(\mathscr{D})} \Big(\int\limits_{\pi^{-1}(\underline{x})} f(\underline{x}, \underline{y}) d^{n_{\underline{y}}}\underline{y} \Big) d^{n_{\underline{x}}}\underline{x}$.
      ii. Prove: $\int\limits_{\pi^{-1}(\underline{x})} f(\underline{x}, \underline{y}) d^{n_{\underline{y}}}\underline{y}$ exists for all $\underline{x} \in \pi(\mathscr{D})$ except for a subset of measure 0.
   (b) Using Fubini prove: if f is C^2 then $\{\partial^2_{ij} f = \partial^2_{ji} f\}_{ij}$.

(5) (a) Compute the integrals:
      i. $\int\limits_1^e dx \int\limits_0^{ln(x)} \frac{dy}{e^y + 1}$
      ii. $\iint\limits_{\mathscr{D}} y dx dy dz, \quad \mathscr{D} = \{\underline{x} \mid\ |x| \leq z,\ 0 \leq z \leq 1,\ x^2 + y^2 + z^2 \leq 4\}$
   (b) Compute $vol_n(P_t)$ of the pyramid $P_t := \{\underline{x} |\ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq t\} \subset \mathbb{R}^n$.
   (c) For any continuous function $[0, 1] \xrightarrow{f} \mathbb{R}^1$ prove:
      i. $\int_{P_t} f(x_n) dx_1 \cdots dx_n = \int\limits_0^t \frac{f(\xi)(t-\xi)^{n-1}}{(n-1)!} d\xi$.
      ii. $\int\limits_{P_t} \prod_{i=1}^{n} f(x_i) dx_1 \dots dx_n = \frac{1}{n!}(\int\limits_0^t f(t) dt)^n$.

(6) We compute the volume of the n-dimensional ball, $Ball^{(n)}_R(0) \subset \mathbb{R}^n$.
   (a) Take the projection $Ball^{(n)}_R(0) \xrightarrow{\pi} Ball^{(2)}_R(0)$, $\pi(\underline{x}) = (x_{n-1}, x_n)$. For each $(x_{n-1}, x_n) \in Ball^{(2)}_R(0)$ verify: $\pi^{-1}(x_{n-1}, x_n) = Ball^{(n-2)}_{\sqrt{1 - x^2_{n-1} - x^2_n}}(0) \times \{(x_{n-1}, x_n)\}$.
   (b) Obtain the recursion $vol_n Ball^{(n)}_R(0) = \frac{2\pi R^2}{n} vol_{n-2} Ball^{(n-2)}_R(0)$ and the formula for $vol_n Ball^{(n)}_R(0)$.
   (c) Compute $\lim\limits_{n\to\infty} vol_n Ball^{(n)}_R(0)$ and $\lim\limits_{n\to\infty} \frac{vol_n Ball^{(n)}_R(0) - vol_n Ball^{(n)}_{R-\epsilon}(0)}{vol_n Ball^{(n)}_R(0)}$ for fixed $R > \epsilon > 0$.