Geometric Calculus 1, 201.1.1031

Homework 13

Fall 2019 (D.Kerner)

(Some hints are on the next page)

(1) (Cavalieri's principle) Suppose $S_1, S_2 \subset \mathbb{R}^n$ admit volume. Take the projection $\mathbb{R}^n_{\underline{x}} \xrightarrow{\pi} \mathbb{R}^k_{\underline{t}}$, $(x_1, \dots, x_n) \to (x_1, \dots, x_k)$ and its restrictions $\pi|_{S_1}, \pi|_{S_2}$. Suppose $vol_{n-k}\big(\pi|^{-1}_{S_1}(\underline{t})\big) = vol_{n-k}\big(\pi|^{-1}_{S_2}(\underline{t})\big)$ for each t. Prove: $vol_n(S_1) = vol(S_2)$.

(2) (a) Compute vol_n of the sets: i. $\{(x, y)|\ (x^2 + y^2)^2 \le 2a^2(x^2 - y^2)\}$
ii. $\{(x, y, z)|\ |x + 2y + 3z| + |2x + 3y + z| + |3x + y + 2z| \le 1\} \subset \mathbb{R}^3$.
iii. $\{\underline{x}|\ \sum_{i=1}^k x_i^2 = 1 = \sum_{i=k+1}^n x_i^2\} \subset \mathbb{R}^n$.

(b) Compute $\iiint_V zye^{x+y^2} dxdydz$, where $V = \big\{0 \le z \le 2,\quad \frac{x}{3} \le z \le \frac{x}{2},\quad \frac{y^2}{4} \le z \le y^2\big\} \subset \mathbb{R}^3$.

(c) Let $Ball^{||*||_p}_R := \{\underline{x}|\ ||\underline{x}||_p < R\} \subset \mathbb{R}^2$. Reduce $vol_2(Ball^{||*||_p}_R)$ to a (possibly improper) integral of the form $\int_0^{2\pi} \cdots d\phi$. Compute it for $p = 1, \frac{1}{2}, \frac{1}{3}$.

(d) ("Because of the symmetry...") Suppose f is integrable on $S \subset \mathbb{R}^n$ and both f, S are invariant under the map $\{x_j \to -x_j\}$, for any j. Prove: $\int_S f d^n\underline{x} = 2^n \cdot \int_{S\cap\{x_1,\dots,x_n>0\}} f d^n\underline{x}$.

(e) Suppose $S \subset \{y > 0\} \subset \mathbb{R}^2_{yz}$ admits vol_2 and V is obtained by the rotation of S around $\hat{z}$-axis. Compute $vol_3(V)$.
i. In particular, compute the volume bounded by the torus, defined in hwk.0, q. 5.e.
ii. Re-compute this by introducing the toric coordinates, parametrizing the torus.

(3) Small things around the proof of the change of variables
(a) Does every open bounded set admit volume? What about compact sets?
(b) Take an open cover of a compact set, $K \subset \cup\mathcal{U}_\alpha \subset \mathbb{R}^n$. Prove: there exists a partition $P = \{\square_i\}$ such that for each i either $\square_i \subset \mathcal{U}_\alpha$ or $\square_i \cap K = \varnothing$.

(4) Let $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{\phi} \mathbb{R}^m$ be C^1, with $m > n$. Prove: $vol_m(\phi(S)) = 0$ for any compact $S \subset \mathcal{U}$.

(5) (a) Let $Box_{\underline{y}} = \prod[a_i, y_i]$ and define $F(\underline{y}) := \int_{Box_{\underline{y}}} f d^n\underline{x}$, for a continuous $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$. Compute F'.

(b) Suppose $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is continuous, define $F(t) := \int_{Ball_t(0)} f(||\underline{x}||)d^n\underline{x}$. Compute $F'(t)$.

(c) For $f = 1$ this can be interpreted as $vol_{n-1}(S^{n-1})$. (In the next semester.)

(6) For a C^1 function $\mathbb{R}^n \supseteq \mathscr{D} \xrightarrow{f} \mathbb{R}^n$ take the set of critical points, $Crit(f) = \{x \in \mathscr{D}|\ rank(f'|_x) < n\}$.
(a) For any $\epsilon > 0$ give an example of surjective function $(0, 1) \xrightarrow{f} \mathbb{R}$ such that $vol_1 Crit(f) = 1 - \epsilon$.
(b) We prove Sard's lemma: *The set of critical values is of measure zero,* $\mu_n\big(f(Crit(f))\big) = 0$.
(i) Verify Sard's lemma for linear maps.
(ii) For any $x \in Crit(f)$ prove: $\lim\limits_{\epsilon\to 0} \frac{vol_n\left(f(Ball_\epsilon(x))\right)}{vol_n Ball_\epsilon(x)} = 0$.
(iii) Finish the proof of Sard's lemma by compact/countable arguments.
(c) The general Sard-Morse theorem generalizes Sard's lemma to C^k maps $\mathbb{R}^n \supseteq \mathscr{D} \xrightarrow{f} \mathbb{R}^m$, with $k = max\{1, n - m + 1\}$.
(d) (i) Let $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ be C^2 and consider the subsets $C_t := \{(x, y)|\ f(x, y) = t\} \subset \mathbb{R}^2$, for $t \in \mathbb{R}$. Prove: for almost all values of t these subsets are smooth curves (or empty).
(ii) $\mathbb{R}^n \supseteq \mathscr{D} \xrightarrow{f} \mathbb{R}$ is called a (C^1) Morse function if all its critical points are non-degenerate, i.e. if $f'|_x = 0$ then $det[f''|_x] \neq 0$. Prove: for any f the function $f(\underline{x}) - \sum a_i x_i$ is Morse for almost all $\underline{a} \in \mathbb{R}^n$ (except of a set of measure zero).
(Because of this some engineers believe that there are no degenerate critical points.)

(a) (For q.3.b.) Consider the function $f(x) = max\{dist(x, \partial\mathcal{U}_\alpha| \ x \in \mathcal{U}_\alpha)\})$
(b) (For q.4.) If Box is "shrunk" by a factor of C then $vol_m(\phi(Box))$ reduces by a factor of C^m.
(c) (For q.6.b.ii) Helpful: use $GL(n, \mathbb{R})$ on $f = (f_1, \ldots, f_n)$ to assume $f'_n|_x = 0$.