Geometric Calculus 1, 201.1.1031

Homework 14

Fall 2019 (D.Kerner)

(1) (a) Verify: finite intersections/unions of exhaustions are exhaustions.
  (b) Given $S \xrightarrow{f} \mathbb{R}_{\geq 0}$, suppose $\lim \int_{S_j} f d^n \underline{x} < \infty$ for a particular exhaustion $\cup S_j = S$.
  Prove: the integral $\int_S f d^n \underline{x}$ converges.

(2) (a) For $\underline{x} \in \mathbb{R}^n$ define $h_k(\underline{x}) = \sum_{i=1}^k x_i^2$. Check the convergence (for $k \leq n$, $\alpha, \beta > 0$):
  i. $\int_{Ball_1(0)} \frac{d^n \underline{x}}{h_k^\alpha ln(1+h_k(\underline{x}))^\beta}$ ii. $\int_{\mathbb{R}^n \setminus Ball_1(0)} \frac{d^n \underline{x}}{h_k^\alpha ln(1+h_k(\underline{x}))^\beta}$.
  (b) Suppose $f, g \geq 0$ and the integrals $\int_{\mathbb{R}^n} f(\underline{x}) d^n \underline{x}$, $\int_{\mathbb{R}^m} g(\underline{y}) d^m \underline{y}$ converge. Prove:
  $\big(\int_{\mathbb{R}^n} f(\underline{x}) d^n \underline{x}\big) \cdot \big(\int_{\mathbb{R}^m} g(\underline{y}) d^m \underline{y}\big) = \int_{\mathbb{R}^{n+m}} f(\underline{x}) g(\underline{y}) d^n \underline{x} d^m \underline{y}$.
  (c) Compute $\int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy$. Deduce $\int_{-\infty}^{\infty} e^{-x^2} dx$.
  (d) Using $\int_{\mathbb{R}^n} e^{-||\underline{x}||^2} d^n \underline{x}$ express $vol_n Ball_1^{(n)}(0)$ via an integral in one variable.
  (e) For which α does $\int\limits_{0 \leq y \leq x^\alpha \leq 1} \frac{dxdy}{x^2+y^2}$ converge?
  (f) Can we paint an infinite wall with a finite amount of ink?

(3) (a) For any sequence of open sets, $X_j \subset (j, j+1)$, construct a continuous function satisfying: $\int_{X_j} f dx = \frac{(-1)^j}{j}$, $f|_{\mathbb{R}^1 \setminus \cup X_j} = 0$. Give two exhaustions of $\cup X_j$ with different limits of $\int f dx$.
  (b) Does $\int_1^\infty cos(x^2) dx$ converge in the sense of Calculus 2? Does $\int_{[1,\infty)} cos(x^2) dx$ converge?
  (c) Compute $\lim\limits_{n \to \infty} \iint\limits_{x^2+y^2 \leq 2\pi n + \pi} sin(x^2+y^2) dxdy$, $\lim\limits_{n \to \infty} \iint\limits_{x^2+y^2 \leq 2\pi n} sin(x^2+y^2) dxdy$.

(4) (a) Assume S admits volume, let S_i be an exhaustion. Prove: $\lim vol_n(S_i) = vol_n(S)$.
  (b) Assume $\int_S f d^n \underline{x}$ converges. Prove: $\int_S f d^n \underline{x} = \lim \int_{S_j} f d^n \underline{x}$.

(5) (a) Find the tangent plane to the graph of $\mathbb{R}^2 \supset \mathscr{D} \xrightarrow{f,g} \mathbb{R}^2$ at the point $(x, y) = (-2, -2)$, where $f(x,y) - g(x,y)^2/2 = x$, $g(x,y) - f(x,y)^2/2 = y$.
  (b) Consider the curve $C = \{f_1(\underline{x}) = 0 = f_2(\underline{x})\} \subset \mathbb{R}^3$, where f_1, f_2 are C^1 and $rank(f'|_{\underline{x}_0}) = 2$. Prove: the tangent line to C at $\underline{x}_0$ is given by $\{\underline{x} | \ \underline{x} - \underline{x}_0 \in ker(f'|_{\underline{x}_0})\}$.

(6) (The joy of bump functions)
  (a) Take a C^∞ function $\mathbb{R} \xrightarrow{\tau} \mathbb{R}_{\geq 0}$ that is flat at 0 and $\tau|_{\mathbb{R} \setminus \{0\}} > 0$. (e.g. $\tau(x) = e^{-\frac{1}{x^2}}$)
  (b) Let $f(x) := \tau(x^2 - 1)$ for $|x| \leq 1$, and 0 on $\mathbb{R}^1 \setminus [-1, 1]$. Verify: $f \in C^\infty(\mathbb{R})$. (A bump function.)
  (c) Construct a monotonic function $g \in C^\infty(\mathbb{R})$ such that $g|_{(-\infty,0]} = 0$ and $g|_{[1,\infty)} = 1$. (e.g. $g(x) := \frac{\int_0^x f(t)dt}{\int_0^1 f(t)dt}$)
  (d) For the following open sets construct a C^∞ function satisfying: $f|_{\mathcal{U}} > 0$, $f|_{\mathbb{R}^n \setminus \mathcal{U}} = 0$.
  i. $\mathcal{U} = Ball_r(a)$ ii. $\mathcal{U} = Int(Box)$ iii. $\mathcal{U} \subset \mathbb{R}^n$ any open set.
  (Useful: open sets have nice coverings by boxes)
  This proves a theorem of Whitney: any closed subset of $\mathbb{R}^n$ is the zero locus of a C^∞ function.
  (e) Let $K \subset \mathcal{U} \subset \mathbb{R}^n$ be a compact inside an open. Construct a C^∞ function f such that $f|_K = 1$ and $f|_{\mathbb{R}^n \setminus \mathcal{U}} = 0$. (Hint: if $f|_K \geq 1$ then you can take $g \circ f$, with g from (c).)

Such bump functions smoothen $\mathbb{1}_K$. They are highly useful in Analysis/Differential Geometry. For geometries over other fields (and Algebra/Arithmetics) one works hard to substitute them.