# Geometric Calculus 1, 201.1.1031

## Homework 2

Fall 2019 (D.Kerner)

(1) (a) Suppose for some $S \subset \mathbb{R}^n$ holds: $\mathbb{R}^n \setminus S$ has no interior points. (Dis)Prove: $\overline{S} = \mathbb{R}^n$, $int(S) \neq \varnothing$.
(b) Suppose for some $S \subset \mathbb{R}^2$ and $x \notin S$ holds: every line through $x$ contains an open segment that contains $x$ and lies fully in $\mathbb{R}^2 \setminus S$. Does this imply $x \notin \overline{S}$? (Hint: no need to look for monsters, consider just parabolas.)
(c) Fix a vector $\vec{v} \in \mathbb{R}^n$ and define the shift $T_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n$, by $\vec{x} \to \vec{x} + \vec{v}$. Prove: if a subset $S \subset \mathbb{R}^n$ is open/closed then $T_{\vec{v}}(S)$ is open/closed.

(2) (a) Take a sequence of points $\{\underline{x}_k\}$ in $\mathbb{R}^n$, here $\underline{x}_k = (x_{1,k}, \dots, x_{n,k})$. Prove:
i. $\underline{x}_k \to \underline{x}_0 \in \mathbb{R}^n$ iff $x_{j,k} \to x_{j,0}$ for any $j$. ii. If $\underline{x}_k \to \underline{x}_0$ then for any subsequence holds: $\underline{x}_{k_j} \to \underline{x}_0$.
(b) Prove: a subset $X \subset \mathbb{R}^n$ is closed iff for any sequence $\underline{x}_k \in X$ the convergence in $\mathbb{R}^n$ implies that in $X$.
(c) A drunken frog jumps in $\mathbb{R}^n$. The direction and the length of each jump are random, but the lengths are bounded, $d_k \leq \frac{1}{k \cdot ln(k+1)}$ (Due to the heavy load of homeworks.) Does the frog necessarily converge to a point? Is the set of its steps necessarily a closed subset of $\mathbb{R}^2$? Can this set have the non-empty interior?

(3) Given a subset $X \subset \mathbb{R}^n$ we have defined the induced topology on $X$.
(a) Describe the open sets on: i. $[a, b] \subset \mathbb{R}^1$, ii. $S^2 \subset \mathbb{R}^3$.
(b) Solve question 8 of homework 1 with $\mathbb{R}^n$ replaced by $X$.

(4) (a) Let $f(x, y) = \begin{cases} y^2/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Prove: for every line $l$ through the origin the restriction $f|_l$ is continuous. Is the function $f(x, y)$ continuous at $(0, 0)$?
(b) Let $f(x, y) = \begin{cases} \frac{e^{-\frac{1}{y^2}}}{x}, & x \cdot y \neq 0 \\ 0, & x \cdot y = 0 \end{cases}$. Prove: for every curve $C = \{y = a \cdot |x|^\beta\}$ the restriction $f|_C$ is continuous. Is the function $f(x, y)$ continuous at $(0, 0)$?

(5) (Dis)Prove the following statements:
(a) $\mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$ is continuous iff the preimage of any closed subset is closed.
(b) $\mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$ is continuous iff the image of any open subset is open.
(c) $\mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$ is continuous iff the image of any closed subset is closed.
(d) $\mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$ is continuous iff its graph is a closed subset, $\Gamma_f \subset \mathbb{R}^{n+m}$.

(6) (a) Take a matrix $A \in Mat_{m \times n}(\mathbb{R})$ and define the map $\mathbb{R}^n \xrightarrow{\phi_A} \mathbb{R}^m$, $\vec{x} \to A\vec{x}$. Is the image of an open/closed subset open/closed? (If not, which additional assumption is needed?)
(b) Let $f(x_1, x_2, x_3, x_4) = sin(x_1 - x_2^3) + e^{x_4 - x_3^4} + tan(x_2) \cdot ln(x_1) - 100$ and consider the subsets:
$X_< := \{\underline{x} \in \mathbb{R}^4 | \ f(\underline{x}) < 0\}$, $X_= := \{\underline{x} \in \mathbb{R}^4 | \ f(\underline{x}) = 0\}$, $X_\leq := \{\underline{x} \in \mathbb{R}^4 | \ f(\underline{x}) \leq 0\}$.
Are they closed/open? Identify the boundaries/closures/interiors.

(7) For $\mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$ and $x_0 \in \overline{\mathscr{D}_f}$ define the limit $\lim\limits_{x \to x_0} f(x)$ in two ways: via point-sequences (by Heine) and via open balls (by Cauchy).
(a) Prove the equivalence of the definitions.
(b) (Sandwich lemma) Suppose the functions $\mathscr{D} \xrightarrow{f,g,h} \mathbb{R}^1$ satisfy: $f(x) \leq g(x) \leq h(x)$. Prove: if $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} h(x) = l$ then $\lim\limits_{x \to x_0} g(x)$ exists and equals to $l$.

(8) (a) (Dis)Prove: finite products, unions, intersections of compact sets (in $\mathbb{R}^n$) are compact.
(b) (Dis)Prove: $S \subset \mathbb{R}^n$ is compact iff all its projections onto coordinate hyperplanes ($\{x_j = 0\} \subset \mathbb{R}^n$) are compact.