Geometric Calculus 1, 201.1.1031

Homework 3

Fall 2019 (D.Kerner)

(1) Check the domain/continuity of the function $f(x,y)=\sqrt{y^2-4}\cdot ln(5-x^2-y^2)$. Draw the level curve at the height 0. Check the existence of the limits: i. $\lim\limits_{(x,y)\to(0,-2)} f(x,y)$ ii. $\lim\limits_{(x,y)\to(0,\sqrt{5})} f(x,y)$ iii. $\lim\limits_{(x,y)\to(1,2)} f(x,y)$.

(2) Define $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ by $f(x,y)=\frac{(x^2+y^4)^3}{1+x^6y^4}$. Prove: $\lim\limits_{|x|\to\infty} f(x,k\cdot x)=\infty$, for any $k\in\mathbb{R}$. Does $\lim\limits_{|(x,y)|\to\infty} f(x,y)$ exist?

(3) In the following cases the domain of definition is not a closed subset of $\mathbb{R}^n$. To which maximal domain can you extend the function in a continuous way?
i. $f(x,y)=x\cdot ln(x^2+3y^2)$ ii. $f(x,y)=y\cdot sin\frac{1}{x}$ iii. $f(x,y)=\frac{x^2y}{x^4+y^2}$ iv. $f(x,y,z)=\frac{sin(x+y+z)-sin(x+y-z)}{z}$.

(4) (a) Compute the repeated limit: $\lim\limits_{m\to\infty}\Big(\lim\limits_{n\to\infty} cos^n(\pi\cdot m!\cdot x)\Big)$. Does the double limit $\lim\limits_{m,n\to\infty} cos^n(\pi\cdot m!\cdot x)$ exist?
(b) Denote the coordinates in $\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}$ by $(\underline{x},\underline{y})$ and consider a function $\mathbb{R}^{n_x+n_y}\xrightarrow{f}\mathbb{R}$. Suppose there exists $\lim\limits_{(\underline{x},\underline{y})\to(\underline{x}_0,\underline{y}_0)} f(\underline{x},\underline{y})$. Suppose for some $\epsilon>0$ holds: for any $(\underline{x},\underline{y}_0)\in Ball_\epsilon(\underline{x}_0,\underline{y}_0)$ exists $\lim\limits_{\underline{y}\to\underline{y}_0} f(\underline{x},\underline{y})$. Prove: $\lim\limits_{(\underline{x},\underline{y})\to(\underline{x}_0,\underline{y}_0)} f(\underline{x},\underline{y})=\lim\limits_{\underline{x}\to\underline{x}_0}\Big(\lim\limits_{\underline{y}\to\underline{y}_0} f(\underline{x},\underline{y})\Big)$.

(5) (a) A sequence of points $\{\underline{x}_k\}$ in $\mathbb{R}^n$ is called a Cauchy sequence if for any ϵ exists N such that $||\underline{x}_k-\underline{x}_{k'}||<\epsilon$ for any $k,k'>N$. Prove: any Cauchy sequence in $\mathbb{R}^n$ converges.
(b) Take a sequence of non-empty compact sets, $K_1\supseteq K_2\supseteq\cdots$. Prove: $\cap K_i\neq\varnothing$. (Does this hold for closed sets? For open sets?)
(c) Suppose $S\subset\mathbb{R}^n$ is non-compact. Show that there exists an unbounded continuous function on S.
(d) Prove: a closed subset of a compact set is compact.
(e) Suppose a function $S_1\xrightarrow{f}S_2$ is continuous, bijective and S_1 is compact. Prove: f^{-1} is also continuous. What can happen when S_1 is non-compact?

(6) Define the distance between the subsets $S_1,S_2\subset\mathbb{R}^n$ by $d(S_1,S_2):=inf\{d(s_1,s_2)|\ s_i\in S_i)$. Prove:
(a) $d(pt,S)=0$ iff $pt\in\overline{S}$. (Give an example with $pt\notin S$.)
(b) If S is closed then there exists $s\in S$ such that $d(pt,S)=d(pt,s)$. (What can happen if S is not closed?)
(c) If $S_1,S_2\subset\mathbb{R}^n$ are bounded then $d(S_1,S_2)=0$ iff $\overline{S_1}\cap\overline{S_2}\neq\varnothing$. (Give an example of bounded sets with $S_1\cap S_2=\varnothing$ but $d(S_1,S_2)=0$. Given an example of unbounded sets with $\overline{S_1}\cap\overline{S_2}=\varnothing$ but $d(S_1,S_2)=0$.)
(d) If S_1,S_2 are compact then exist $s_1\in S_1$, $s_2\in S_2$ such that $d(s_1,s_2)=d(S_1,S_2)$.
(e) Suppose for $S_1,S_2\subset\mathbb{R}^n$ holds: $d(S_1,S_2)>0$. Then there exists a continuous function $\mathbb{R}^n\xrightarrow{f}\mathbb{R}$ such that $f|_{S_1}=0$ and $f|_{S_2}=1$. (For example, one can define $f(x):=2\frac{d(x,S_1)}{d(S_1,S_2)}$, if $d(x,S_1)\leq\frac{d(S_1,S_2)}{2}$, and $f(x)=1$ otherwise. Check the continuity.)
This is a very strong separation property. You will see more of this kind in the course "Introduction to Topology". Conclude: for any compact sets with $\overline{S_1}\cap\overline{S_2}=\varnothing$ there exist open neighborhoods, $S_i\subset\mathcal{U}_i\subset\mathbb{R}^n$ such that $d(\mathcal{U}_1,\mathcal{U}_2)>0$. (In particular, $\overline{\mathcal{U}_1}\cap\overline{\mathcal{U}_2}=\varnothing$.)

(7) (a) Which of the following sets (defined by parametrization) are path-connected?
i. $\{(\frac{1}{t^2},\frac{1}{t^3})|\ t\in(1,\infty)\}\coprod\{0,0\}$ ii. $\{r=\frac{1}{1+\phi}|\ \phi\in[1,\infty)\}\coprod\{(0,0)\}$ (r,ϕ are the polar coordinates in $\mathbb{R}^2$).
(b) Suppose the subsets $\{X_\alpha\}$ of $\mathbb{R}^n$ are path-connected, and $\cap X_\alpha\neq\varnothing$. Prove: $\cup X_\alpha$ is path-connected. (Here the collection is not necessariy finite)
(c) Let $\{X_i\}$ be a finite collection of path-connected sets. Prove: $\prod X_i$ is path-connected.
(d) Prove: path-connectedness is preserved under linear transformations and shifts of $\mathbb{R}^n$.
(e) Prove: $S^{n-1}:=\{\underline{x}|\ |\underline{x}|=r\}\subset\mathbb{R}^{n>1}$ is path connected. Do this in as many ways as you can, e.g.: by considering the polar coordinates in $\mathbb{R}^n$; by presenting S^{n-1} as the union of two graphs of continuous functions; by intersecting S^{n-1} with a hyperplane.
(f) Let $X\subset\mathbb{R}^n$ and suppose $a\in int(X)$, $b\in int(\mathbb{R}^n\setminus\overline{X})$. Prove: any path from a to b intersects ∂X.