Geometric Calculus 1, 201.1.1031

Homework 4

Fall 2019 (D.Kerner)

(1) Check the path-connectedness of the sets:

i. $\{\sum \lambda_i x_i^{n_i} = 1\} \subset \mathbb{R}^n$ (here $\{\lambda_i \in \mathbb{R}\}$ and $\{n_i \in \mathbb{N}\}$ are fixed) ii. $\{x_1(sin(x_2^3 - x_3^2) - x_4^4) = 1\} \subset \mathbb{R}^4$

(2) Using the standard inner product on $Mat_{m\times n}(\mathbb{R})$, $\langle A, B\rangle = trace(A \cdot B^t)$, we identify $Mat_{m\times n}(\mathbb{R})$ with $\mathbb{R}^{mn}$. Thus we can define open/closed balls and open/closed/compact/bounded/path-connected subsets of $Mat_{m\times n}(\mathbb{R})$.

(a) Describe (as explicitly as possible) $Ball_r(0)$ and the standard sphere of radius 1 in $Mat_{2\times 2}(\mathbb{R})$.

(b) Prove that the trace and the determinant are continuous functions, $Mat_{n\times n}(\mathbb{R}) \to \mathbb{R}$.

(c) More generally, prove that all the coefficients of the characteristic polynomial of A depend continuously on A.

(d) Consider the matrix product, $Mat_{m\times n}(\mathbb{R}) \times Mat_{n\times k}(\mathbb{R}) \to Mat_{m\times k}(\mathbb{R})$, $(A, B) \to A \cdot B$. Prove: this is a continuous function. Prove: the inverse of a matrix, $GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $A \to A^{-1}$, is a continuous function.

(e) Let $SL(n, \mathbb{R}) := \{A|\ det(A) = 1\}$, $O(n, \mathbb{R}) := \{A|\ A \cdot A^t = \mathbb{1}\}$, $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap SL(n, \mathbb{R})$.
Prove: $O(n, \mathbb{R})$ sits inside the sphere of radius $\sqrt{n}$ centered at $\mathbb{O} \in Mat_{n\times n}(\mathbb{R})$. Is $SL(n, \mathbb{R})$ bounded?
Which of the sets $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$ are open/closed/compact?

(f) Prove: $O(n, \mathbb{R})$ is not path-connected. Is $GL(n, \mathbb{R})$ path-connected?

(g) Let $X_{diag} \subset Mat_{n\times n}(\mathbb{R})$ be the subset of all the matrices that are diagonalizable over $\mathbb{C}$. (i.e. $U \cdot A \cdot U^{-1}$ is diagonal for some $U \in GL(n, \mathbb{C})$) Prove: any matrix whose eigenvalues are pairwise distinct complex numbers belongs to $\underline{int}(X_{diag})$.

(h) Conclude: $\overline{X_{diag}} = Mat_{n\times n}(\mathbb{R})$ and $int(Mat_{n\times n}(\mathbb{R}) \setminus X_{diag}) = \varnothing$. (Because of this many engineers claim "Any matrix in real life is diagonalizable".)

(i) Is $Mat_{n\times n}(\mathbb{R}) \setminus X_{diag}$ a closed subset of $Mat_{n\times n}(\mathbb{R})$? (Hint: look at $Mat_{2\times 2}(\mathbb{R})$)

(3) Define the polar coordinates in $\mathbb{R}^n$ by $r = \sqrt{\sum x_i^2}$, $\left\{\phi_j = arccos\frac{x_j}{\sqrt{\sum_{i=1}^{j} x_i^2}} \in [0, \pi]\right\}_{j=3,\dots,n}$, while $\phi_2 \in [0, 2\pi)$ is determined by $\phi_2 = arccos\frac{x_1}{\sqrt{x_1^2+x_2^2}} = arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}$.

(a) Verify that for $n = 2, 3$ we get the ordinary polar coordinates in $\mathbb{R}^2$, $\mathbb{R}^3$. Why do we need two expressions for the angle ϕ_2?

(b) Give the geometric definition of all the angles ϕ_j, $j = 2, \dots, n$.

(c) Write the explicit formulas for the map $(r, \underline{\phi}) \to \underline{x}$.

(d) Verify that the domain of the coordinate change $\underline{x} \to (r, \underline{\phi})$ is $\mathbb{R}^n \setminus \{x_1^2 + x_2^2 = 0\}$ while it image is $\mathbb{R}_{>0} \times (0, \pi)^{n-2} \times [0, 2\pi)$.

(e) Verify that the map $\mathbb{R}_{>0} \times (0, \pi)^{n-2} \times [0, 2\pi) \ni (r, \underline{\phi}) \to \underline{x} \in \mathbb{R}^n \setminus \{x_1^2 + x_2^2 = 0\}$ is continuous and bijective. Is its inverse continuous?

(f) Using polar coordinates prove (again) that the sphere $S^{n-1} \subset \mathbb{R}^n$ is compact and path-connected.

(4) (a) We have defined the norms $||*||_p$ on $\mathbb{R}^n$, for $1 \le p \le \infty$. Verify that these are indeed norms. Draw $Ball_1^{||*||_p}(0) \subset \mathbb{R}^2$.

(b) Prove: $\lim\limits_{p\to\infty} ||x||_p = ||x||_\infty$ for any $x \in \mathbb{R}^n$.

(c) Let $C[a, b]$ be the space of functions continuous on $[a, b] \subset \mathbb{R}^1$. Prove that $||f||_p := \sqrt[p]{\int\limits_a^b |f(x)|^p dx}$ defines a norm on $C[a, b]$, for any $1 \le p \le \infty$. (Hint: Hölder and Minkowski)

(d) Prove: $\lim\limits_{p\to\infty} ||f||_p = ||f||_\infty$ for any $f \in C[a, b]$.

(e) Let $|| * ||_1$, $|| * ||_2$ be two equivalent norms on a vector space $V_\mathbb{R}$ (not necessarily of finite dimension). Prove:

(i) $S \subseteq V$ is open/closed/compact/path-connected w.r.t. $|| * ||_1$ iff it is w.r.t. $|| * ||_2$.

(ii) $int^{||*||_1} S = int^{||*||_2} S$, $\overline{S}^{||*||_1} = \overline{S}^{||*||_2}$. (The interior and the closure w.r.t. these norms)

(iii) $V \supseteq \mathscr{D}_f \overset{f}{\to} \mathbb{R}^m$ is $|| * ||_1$-continuous iff it is $|| * ||_2$-continuous.

(iv) $(V, || * ||_1)$ is complete iff $(V, || * ||_2)$ is complete. ($(V, || * ||)$ is complete if every Cauchy sequence converges)

(f) We have skipped some steps in the proof of "All the norms are equivalent on $\mathbb{R}^n$". Fill in all the details.