Geometric Calculus 1, 201.1.1031

Homework 5

Fall 2019 (D.Kerner)

(1) (a) Give an example of a non-complete metric space and a contractive map without base points.
(b) Can the assumption on the contraction map in the fixed point theorem be weakened to $d(f(x), f(y)) < d(x, y)$?

(2) (a) The following functions are not defined on the whole $\mathbb{R}^2$. To which domain can they be extended in a C^0/differentiable/C^1-way?
i. $f(x,y) = \frac{xy^2}{x^2+y^2}$ ii. $f(x,y) = \frac{ln|\frac{x}{y}|}{x-y}$ iii. $f(x,y) = x \cdot sin(\frac{y}{\sqrt{|x|}})$.
(b) Compute f' in the following cases: i. $\mathbb{R}^3 \times \mathbb{R}^3 \stackrel{f}{\to} \mathbb{R}^3$, $f(\vec{v}, \vec{u}) = \vec{v} \times \vec{u}$ (vector product)
ii. The transition to the polar coordinates in $\mathbb{R}^2$ (and the inverse transition).
iii. $\mathbb{R}^2 \stackrel{f}{\to} \mathbb{R}^2$, $f = \underbrace{g \circ g \circ \ldots \circ g}_{n\ times}$, where $g(x,y) = (x^2 - y^2, 2xy)$, compute at the point $(1,0)$.
iv. $\mathbb{R}^n \times \mathbb{R}^n \stackrel{f}{\to} \mathbb{R}^n$, $f(\vec{v}, \vec{u}) = \vec{v} + \vec{u}$. v. $\mathbb{R}^n \stackrel{f}{\to} \mathbb{R}^1$, $f(\underline{x}) = \underline{x} \cdot A \cdot \underline{x}^t$, for some $A^t = A \in Mat_{n\times n}(\mathbb{R})$.
vi. $Mat_{n\times n}(\mathbb{R}) \ni X \to X \cdot A \cdot X \in Mat_{n\times n}(\mathbb{R})$, for a fixed $A \in Mat_{n\times n}(\mathbb{R})$
(c) Suppose $\mathbb{R}^n \supseteq \mathscr{D} \stackrel{f,g}{\to} \mathbb{R}^1$ are differentiable. Express $(f \cdot g)'$, $(\frac{f}{g})'$ via f', g'.
(d) Prove: any function $\mathbb{R}^n \stackrel{f}{\to} \mathbb{R}^m$ that satisfies $|f(\underline{x})| \le |\underline{x}|^{1+\epsilon}$, for some $\epsilon > 0$, in $Ball_\delta(0)$, is differentiable at 0.
(e) Suppose $\mathbb{R}^n \supseteq \mathscr{D}_f \stackrel{f}{\to} \mathbb{R}^n$ is invertible and both f, f^{-1} are differentiable. Prove: $f'|_{\underline{x}_0} \cdot (f^{-1})'|_{f(\underline{x}_0)} = \mathbb{1}$.
(f) We have proved that C^1 implies differentiability for $n = 2$. Extend this to the general case.

(3) (a) Suppose $\mathbb{R}^n \stackrel{f}{\to} \mathbb{R}^1$ is differentiable and homogeneous of order p, i.e. $f(t \cdot \underline{x}) = t^p \cdot f(\underline{x})$ for any $t \in \mathbb{R}$. Prove: $\sum x_i \partial_i f = p \cdot f$.
(b) Find all the points of $\mathbb{R}^n$ at which the directional derivative of $f(\underline{x}) = \sum_{j=1}^{n} sin^2(x_j + a_j)$ in the direction $(1, 2, \ldots, n)$ achieves its maximum. (Here $\{a_j\}$ are constants.)
(c) Compute the directional derivative of $f(x,y) = \begin{cases} \frac{y^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $(0,0)$ in the direction $\vec{v} \in \mathbb{R}^2$.
(What would you expect when presenting this function as $r \cdot \frac{sin^2(\theta)}{cos(\theta)}$?)

(4) (a) An inkspot moves along the surface $\{z = ax^2 + by^2\} \subset \mathbb{R}^3$, $a, b > 0$. At each moment the velocity of the inkspot is in the direction of the steepest descent. Find the equation of the path.
(b) A point moves along the standard sphere $S^{n-1} \subset \mathbb{R}^n$, and its coordinates, $\mathbb{R}^1 \stackrel{\underline{x}(t)}{\to} S^{n-1}$, depend differentiably on time. Prove: at each moment the velocity of the point, $\frac{d\underline{x}(t)}{dt}$, is orthogonal to the vector $\underline{x}(t)$. Prove: the acceleration, $\frac{d^2\underline{x}}{dt^2}$, points into $Ball_1(0)$. (wiki: Centrifugal Force)

(5) (a) Suppose $\mathcal{U} \subset \mathbb{R}^n$ is open and path-connected. Prove: for any two points of $\mathcal{U}$ there exists a path $[0,1] \stackrel{\underline{x}(t)}{\to} \mathcal{U}$, such that $\underline{x}(t)$ is injective (i.e. the path is non-self-intersecting), differentiable, and $\frac{d\underline{x}}{dt} \neq 0$ for $t \in (0,1)$.
(b) Let $\mathscr{D}_f \stackrel{f}{\to} \mathbb{R}^m$ be differentiable, with $\mathscr{D}_f \subset \mathbb{R}^n$ open and path-connected. (Dis)prove: $f = const$ iff $f' = 0$ on $\mathscr{D}_f$.